

Guided Study Program in System Dynamics

System Dynamics in Education Project
System Dynamics Group
MIT Sloan School of Management¹

Solutions to Assignment #12
January 22, 1999

Reading Assignment:

Please refer to Road Maps 5: A Guide to Learning System Dynamics (D-4505-4) and read the following paper from Road Maps 5:

- Generic Structures: S-Shaped Growth, by Terri Duhon and Marc Glick (D-4432)

Please read the following:

- Principles of Systems,² by Jay W. Forrester, Chapter 3

Please refer to Road Maps 7: A Guide to Learning System Dynamics (D-4507-1) and read the following paper from Road Maps 7:

- Graphical Integration Exercises Part 4: Reverse Graphical Integration, by Laughton Stanley (D-4603)

Exercises:

1. Generic Structures: S-shaped growth

After reading this paper and doing all the included exercises, answer the questions below:

A. This paper introduces two generic structures for S-shaped growth and studies the types of behavior that can result from such structures. Refer to Figure 7b on Page 13. Build the “New Product Life Cycle” model in Vensim PLE by following the equations for the Epidemic model in Appendix A, pages 21-23. In your assignment solutions document, please include the model diagram and documented equations.

Use the following values for the constants:

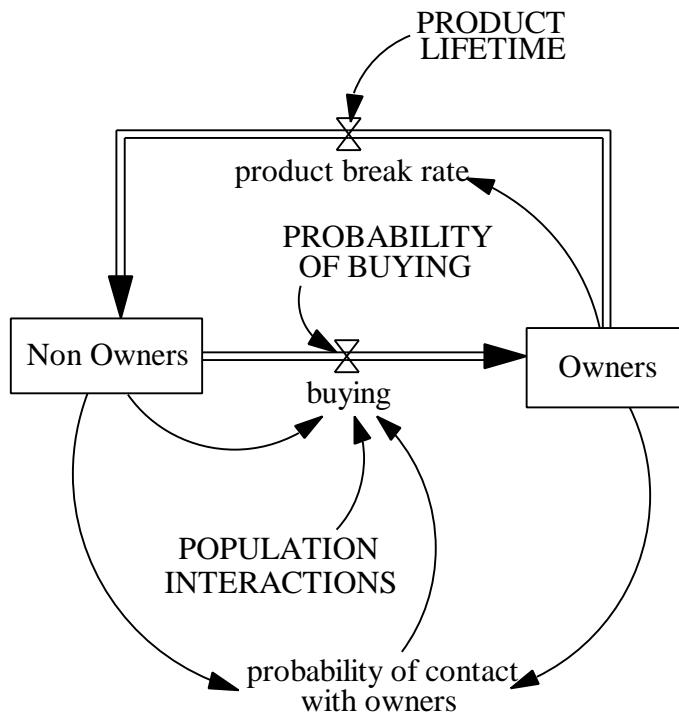
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² Forrester, Jay W., 1968. *Principles of Systems*, (2nd. ed.). Waltham, MA: Pegasus Communications. 391 pp.

*product lifetime = 2 Years
 probability of buying = 0.25
 population interactions = 16 / Year
 There are 2000 people in the system.*

As in the paper, the model looks as follows (we changed the flow from “buyers” to “buying” because “buyers” sounds like its units are “people,” not “people per year”):

Model diagram:



Model equations:

$\text{buying} = \text{POPULATION INTERACTIONS} * \text{Non Owners} * \text{PROBABILITY OF BUYING} * \text{probability of contact with owners}$
 Units: people/Year
 The number of people per year who become new owners by buying the product.

$\text{Non Owners} = \text{INTEG}(\text{product break rate} - \text{buying}, 1998)$
 Units: people
 The number of people at any point in time who do not own the new product.

$\text{Owners} = \text{INTEG}(\text{buying} - \text{product break rate}, 2)$
 Units: people
 The number of people at any point in time who own the new product.

POPULATION INTERACTIONS = 16

Units: 1/Year

The number of interactions per year between owners and non-owners.

PROBABILITY OF BUYING = 0.25

Units: dimensionless

The probability that a non-owner buys the new product and thus becomes an owner.

probability of contact with owners = Owners / (Non Owners + Owners)

Units: dimensionless

The probability that a non-owner comes into contact with an owner equals the number of owners divided by the total population. The units of people/people give us a dimensionless probability for non-owners to come into contact with owners.

product break rate = Owners / PRODUCT LIFETIME

Units: people/Year

The number of people who cease to be product owners each year due to product breakage or wear.

PRODUCT LIFETIME = 2

Units: Year

The number of years a product lasts before it wears out or breaks.

B. What types of behavior can the New Product Life Cycle exhibit? Why?

The model can exhibit three different types of behavior:

- equilibrium: If there are no “Owners” to influence the “Non Owners,” the “probability of contact with owners” is zero, so the “buying” rate is zero. Also, if no one owns the product, no one discards the product, so the “discarding” rate also remains at zero. Without any net flow, the stocks do not change.
The system is also at equilibrium if the two flows are initially equal; that is, if “Owners” discard the product as fast as “Non Owners” buy it.
- asymptotic behavior: When the initial number of “Owners” is larger than the equilibrium value, the stock “Owners” exhibits asymptotic decline towards equilibrium. Asymptotic growth towards equilibrium occurs when the initial value of “Owners” is lower than the equilibrium value, but large enough so that the product’s fixed market size of 2000 people prevents the stock of “Owners” from growing exponentially. Because asymptotic behavior is stabilizing, in both of these cases the system continues to be dominated by negative feedback throughout the simulation.
- S-shaped behavior: The system exhibits S-shaped behavior when the initial number of “Owners” is small. The number of “Owners” grows exponentially from the

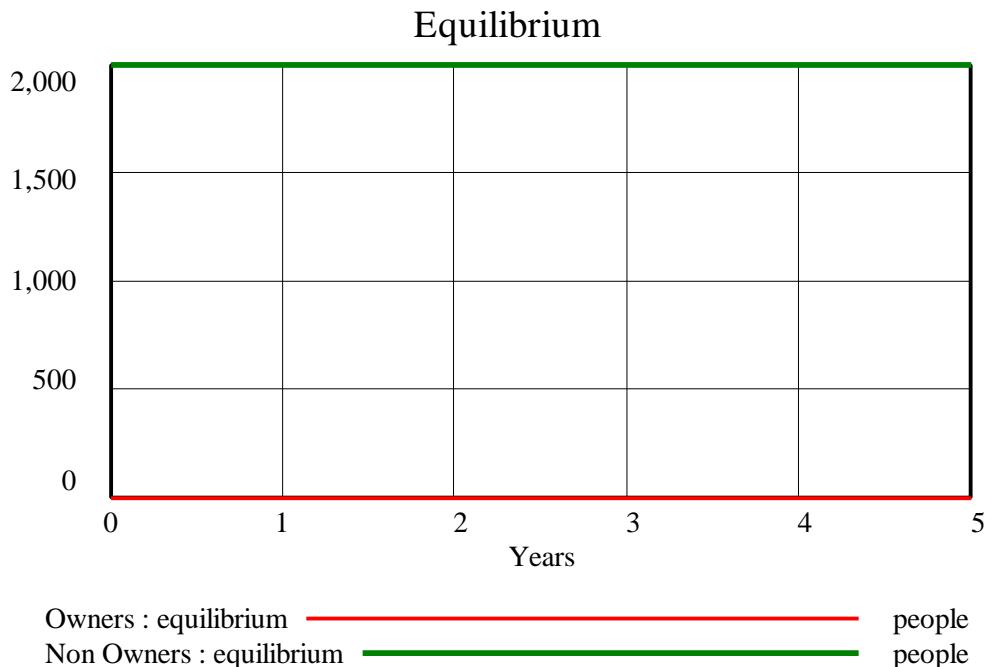
beginning because a large initial number of “Non Owners” drives up the rate at which people buy the product. As the number of “Owners” gets larger, the system shifts to negative feedback loop dominance as the fixed market size limits the growth of “Owners.”

C. For each type of possible system behavior mentioned in part B, find a pair of initial stock values for “Owners” and “Non Owners” that gives such behavior. Remember that the sum of the stocks must always equal 2000 people. In your assignment solutions document, include graphs of behavior for each pair of initial stock values.

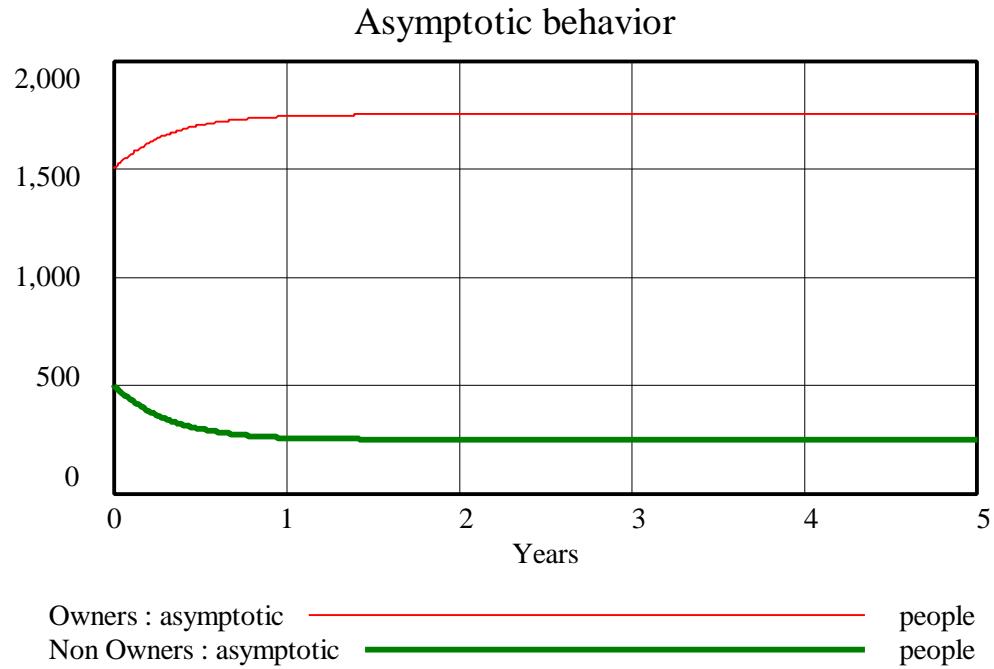
Example: When initial value of “Owners” is X, and initial value of “Non Owners” is (2000-X), the system exhibits _____ behavior as shown in the following graph.

You can find the answers to this question by plugging in any numbers for the initial stock values and simulating the model to see what type of behavior results. There are countless pairs of initial stock values summing to 2000 that will result in asymptotic or S-shaped behavior. The following are some combinations that result in equilibrium, asymptotic, and S-shaped behavior:

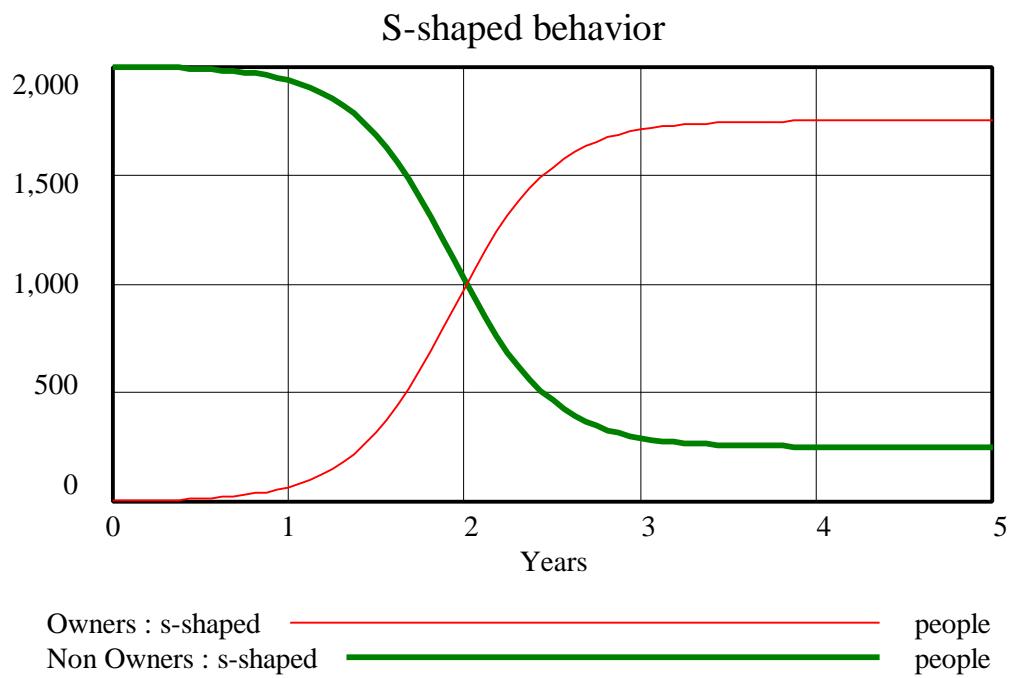
When the initial value of “Owners” is 0, and the initial value of “Non Owners” is 2000, the system is in equilibrium, as shown in the following graph:



When the initial value of “Owners” is 1500, and the initial value of “Non Owners” is 500, the system exhibits asymptotic behavior, as shown in the following graph:



When the initial value of “Owners” is 2, and the initial value of “Non Owners” is 1998, the system exhibits S-shaped behavior, as shown in the graph below:



D. Find the ranges of stock values for which each type of behavior is exhibited. Explain your answer.

Example: When initial value of Owners is between Y and Z, and initial value of Non-Owners is between (2000-Y) and (2000-Z), the system exhibits _____ behavior.

Hint: To answer this question, it is sufficient to consider only the behavior of the stock “Owners.” Looking only at the behavior of a stock, however, may sometimes be misleading. Instead, you may want to look at the behavior of the net flow into the stock, the difference between the inflow and the outflow. What behavior does the stock exhibit when the net flow is increasing? What behavior does the stock exhibit when the net flow is decreasing?

In this exercise, we are looking for ranges of initial stock values that produce equilibrium, asymptotic, and S-shaped system behavior. Because the number of “Owners” and “Non-Owners” always adds up to 2000, we can simply look for the range of initial values for one of the stocks (“Owners,” for example), and simply subtract from 2000 to get the range for the other stock (“Non Owners”).

Finding the initial stock values that produce equilibrium: One way for the system to never change, as we mentioned before, is when there are no “Owners” to interact with and persuade “Non Owners” to purchase the product.

$$\text{Owners} = 0$$

$$\text{Non-Owners} = 2000$$

Note that if all 2000 people were “Owners,” we would not see the same non-changing behavior. Because product breaking depends only on the “Owners” stock and “buying” is initially zero, the “Non Owners” stock would increase.

The second way for the system to never change is for the initial values of the two stocks to be such that the two flows start out at the same value and never veer from that value. A simple calculation reveals the initial values needed to obtain equal flows:

$$\text{buying} = \text{product break rate}$$

$$\begin{aligned} \text{Non Owners} * \text{POPULATION INTERACTIONS} * \text{PROBABILITY OF} \\ \text{BUYING} * \text{probability of contact with owners} &= \text{Owners} / \text{PRODUCT} \\ &\text{LIFETIME} \end{aligned}$$

$$\text{Non Owners} * 16 * 0.25 * \text{Owners} / (\text{Non Owners} + \text{Owners}) = \text{Owners} / 2$$

$$\text{Non Owners} * 4 / (\text{Non Owners} + \text{Owners}) = 0.5$$

$$\text{Non Owners} * 4 = 0.5 * (\text{Non Owners} + \text{Owners})$$

$$\text{Non Owners} * 4 = 0.5 * 2000$$

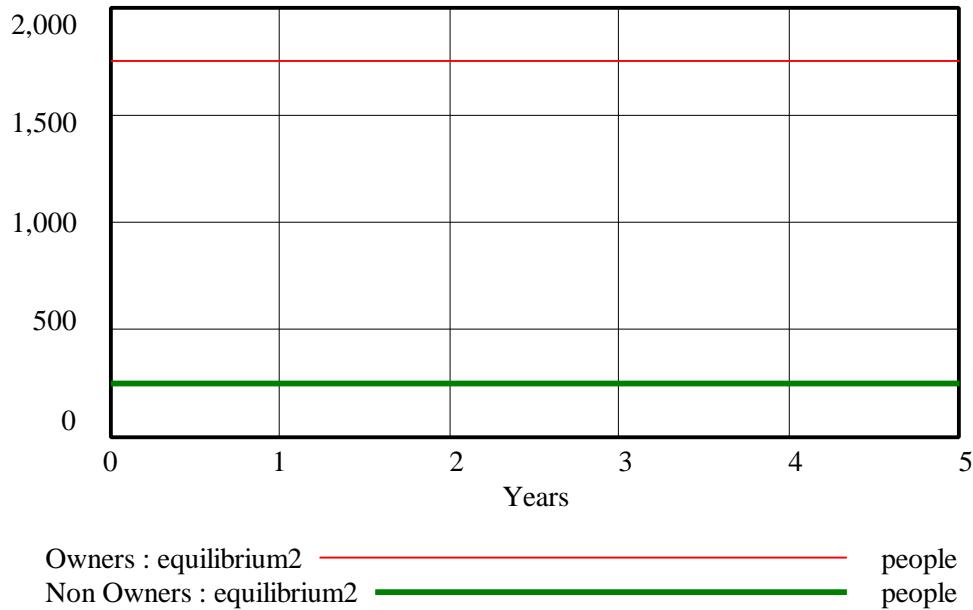
$$\text{Non Owners} = 1000 / 4$$

$$\text{Non Owners} = 250$$

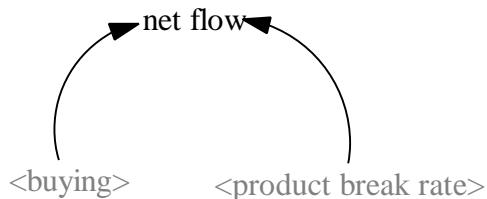
$$\text{Owners} = 2000 - 250 = 1750$$

The graph below shows an equilibrium state when “Owners” = 1750 and “Non Owners” = 250:

Equilibrium with Owners = 1750 and Non Owners = 250



Finding the range of initial stock values to produce S-shaped behavior: S-shaped behavior occurs when the initial stock values are such that positive feedback dominates the system initially, and then shifts to negative feedback. In other words, a system produces S-shaped behavior if the net flow into a stock is initially increasing in magnitude, and at some point, starts decreasing in magnitude but remains either positive or negative throughout the simulation. You can visualize the behavior of the net flow to the “Owners” stock by making the following addition to your model:



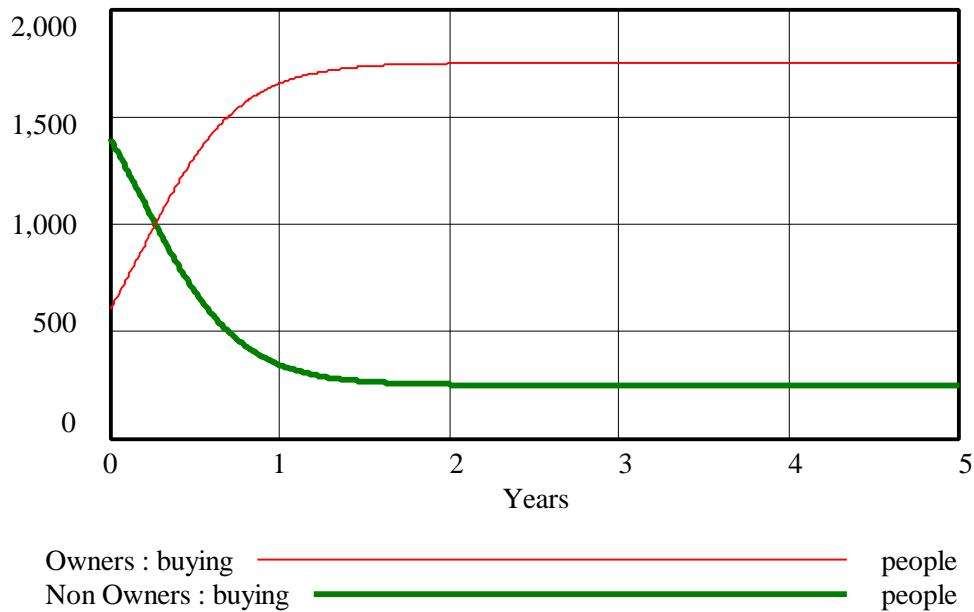
$$\text{net flow} = \text{buying} - \text{product break rate}$$

Units: people/Year

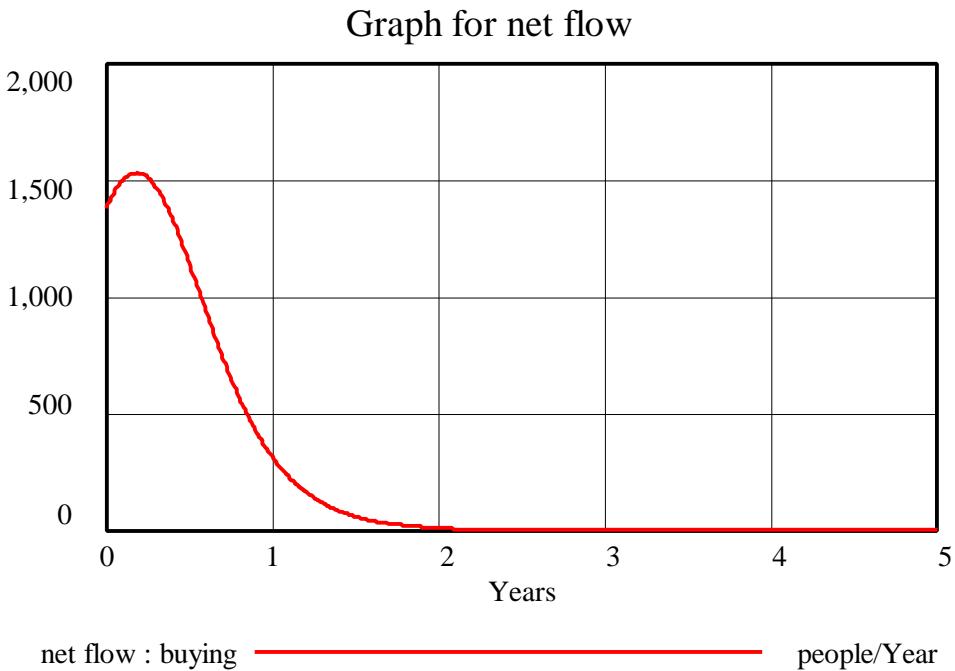
The net flow into the “Owners” stock at any point in time.

Adding the “net flow” to the stock “Owners” (the “Non Owners” net flow is just the negative of the “Owners” net flow) helps one distinguish between S-shaped and asymptotic behavior. For example, if the initial value of “Owners” equals 600, and the initial value of “Non Owners” is 1400, the stock behavior looks like this:

S-shaped or asymptotic behavior?

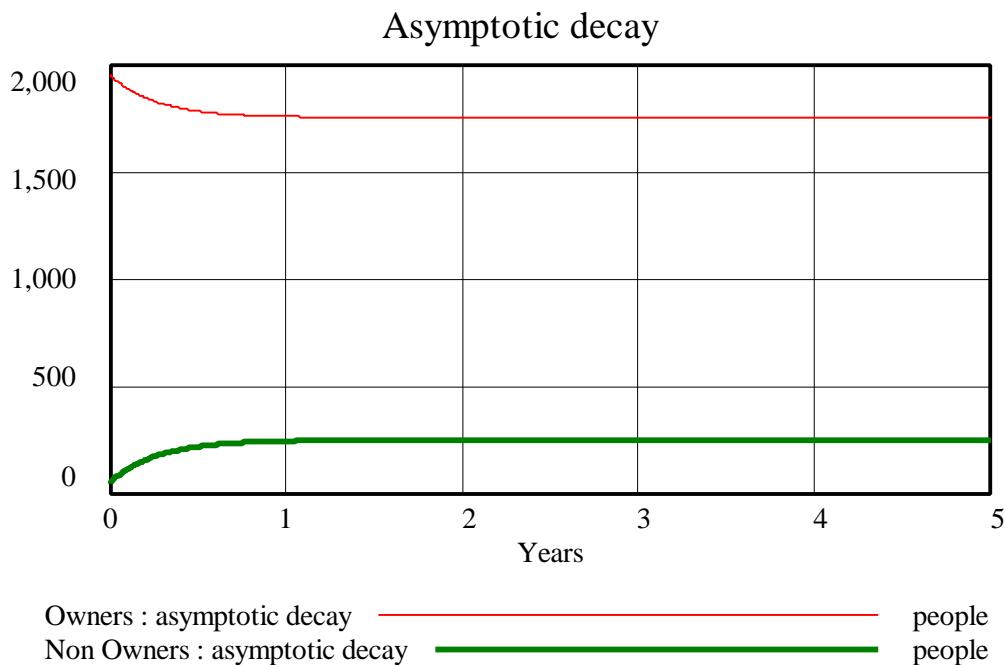


As you can see, it is difficult to tell whether the behavior is pure asymptotic, or whether it is briefly exponential in the beginning of the simulation. Therefore, we look at the behavior of “net flow”:



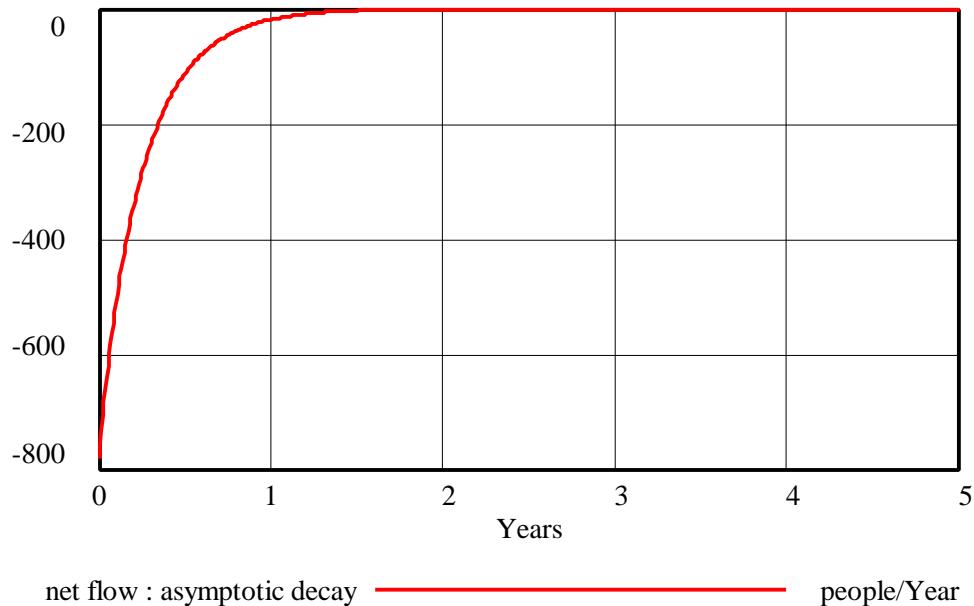
The “net flow” graph clearly shows an initial increase, so we know that these particular initial values do produce S-shaped behavior. From the trial run, we can tell that any initial value of “Owners” below 600 (but greater than 0) will also result in S-shaped growth because the system will start out even farther from its equilibrium. We do not yet know, however, the maximum initial value of “Owners” that will result in S-shaped growth. To find this value, we would like to get a better idea of the range of initial values producing asymptotic behavior.

Finding the ranges of initial stock values producing asymptotic growth and decay: If the initial number of “Owners” is above its equilibrium value of 1750 people, the stock will decay to its equilibrium value asymptotically. The following graph shows asymptotic decay to equilibrium when initially “Owners” = 1950 and “Non Owners” = 50:



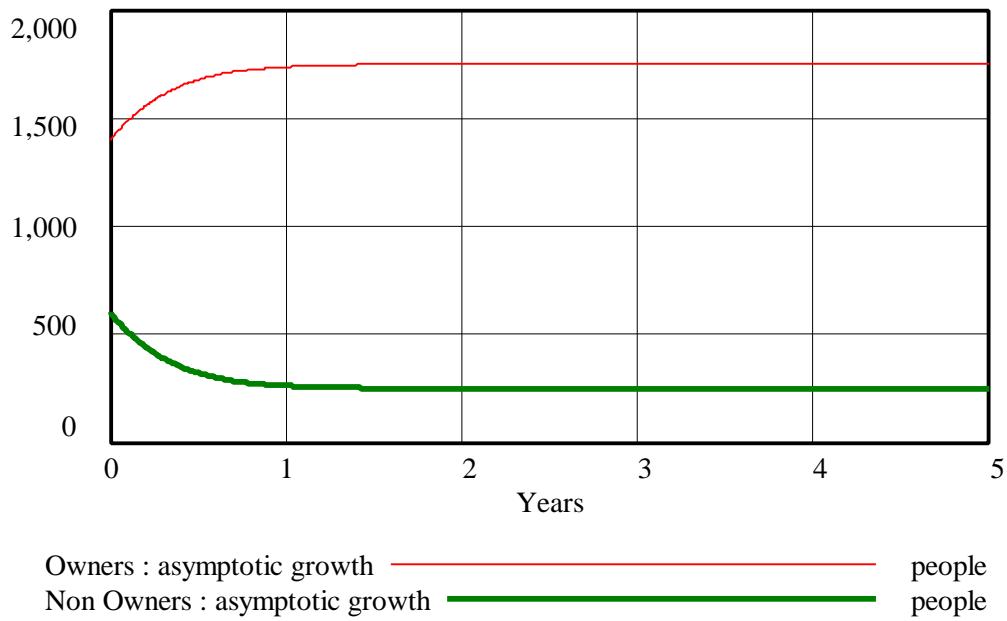
The corresponding graph of “net flow” into the “Owners” stock is negative and decreasing in magnitude:

Graph for net flow for asymptotic decay

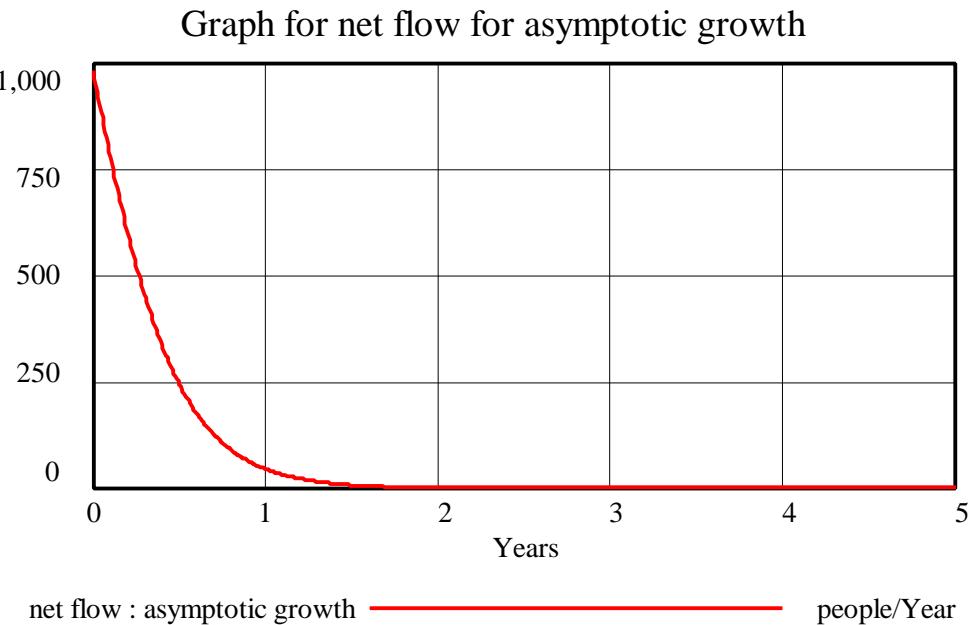


For certain values of “Owners” lower than 1750, the system exhibits asymptotic growth to equilibrium. The following graph shows the stock behavior when the initial value of “Owners” is 1400, and the initial value of “Non Owners” is 600:

Asymptotic growth

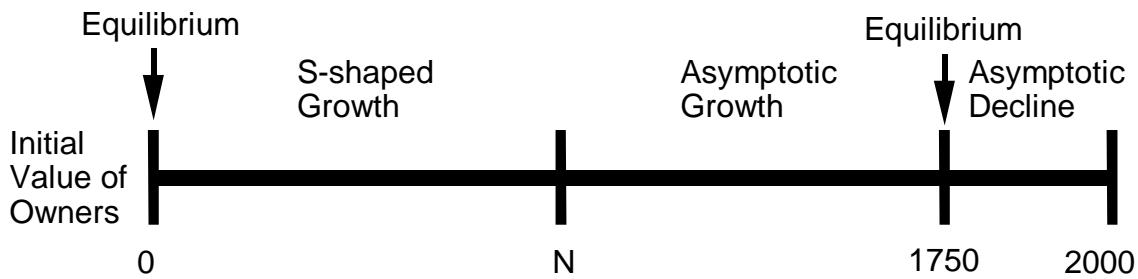


The corresponding graph of “net flow” into the “Owners” stock is positive and decreasing in magnitude:



To recap, we know that for initial values of “Owners” from 1750 to 2000, the stock exhibits asymptotic decay. We also know that for some range of initial values for “Owners” less than 1750, the stock exhibits asymptotic growth. We want to find the minimum initial value of “Owners” resulting in asymptotic growth, which is also the maximum initial value of “Owners” resulting in S-shaped growth.

Finding the initial stock value cutoff between S-shaped growth and asymptotic growth:
 By this point, we have a rough idea of the range of initial values and their corresponding system behaviors. If the initial value of “Owners” is small, the stock will likely experience exponential growth because it is still far from equilibrium. As the number of “Owners” increases, it will reach some value “N” where the negative feedback within the system dominates, and there is no room for exponential growth. Past that value “N,” stock behavior will be either asymptotic growth or decline, depending on the stock value’s position relative to its equilibrium point. The following figure illustrates the ranges:



We know that for an initial value of Owners between 1750 and 2000, Owners will asymptotically decline to equilibrium. We also know that if Owners = 0 or Owners = 1750, the system is in equilibrium. The tricky part now is to figure out at which point “N” the system’s behavior shifts from S-shaped growth to asymptotic growth.

S-shaped growth happens when first, the system grows exponentially, and then shifts into asymptotic growth. Exponential growth happens when the “net flow” into the “Owners” stock (“buying” – “product break rate”) increases. In other words, not only must the “net flow” be positive (that is, “buying” must be greater than “product break rate”), but the rate of “buying” must be growing faster than the “product break rate.” As soon as the “product break rate” starts increasing faster than the rate of “buying,” the “net flow” into “Owners” will start decreasing, and this occurs exactly when the stock of “Owners” is at the value N.

There are two ways of locating the value of N:

1. Trial and error method: Plug in initial values of “Owners” between 0 and 1750, remembering to set “Non Owners” equal to 2000 – “Owners.” For each trial, look at the “net flow” graph to see if the “net flow” increases at all. If so, the system is generating S-shaped behavior. If not, the system behavior is asymptotic. Keep trying different numbers until you locate the place where the system shifts from S-shaped to asymptotic behavior. In order to determine “N” with the utmost accuracy, you may want to decrease the time step or look at a table of values of the “net flow” because it may be difficult to tell whether it is increasing on a graph. You can also try to “zoom in” on the left side of the graph by selecting the appropriate time range for the graph display.
2. Differential equations method: You need to have an understanding of calculus in order to mathematically derive the point “N.” For those who are interested, the solution is attached to the end of this document in the appendix.

By either one of the methods, you will find that the value of “N” is 875, the midpoint between the two equilibrium values. Thus we can conclude the following:

- For Owners = 0 or Owners = 1750, the system is in equilibrium
- For $0 < \text{Owners} \leq 875$, the stock generates S-shaped growth to equilibrium
- For $875 < \text{Owners} \leq 1750$, the stock asymptotically grows to equilibrium
- For $1750 < \text{Owners} \leq 2000$, the stock asymptotically decays to equilibrium

2. Principles of Systems

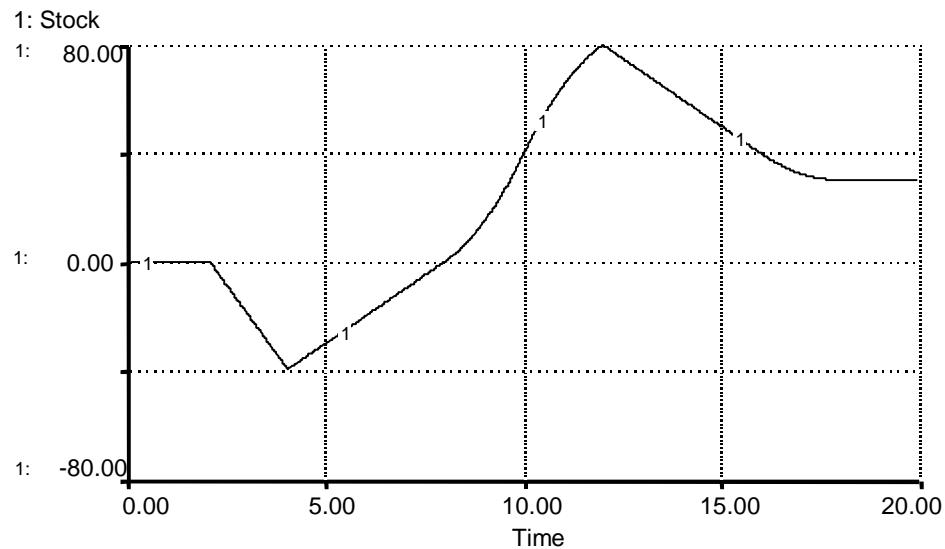
Please read chapter 3 of Principles of Systems and do the workbook exercises for this chapter (located at the end of the book). Let us know if you have problems or questions

about the material in this reading. You do not need to submit anything for this reading assignment.

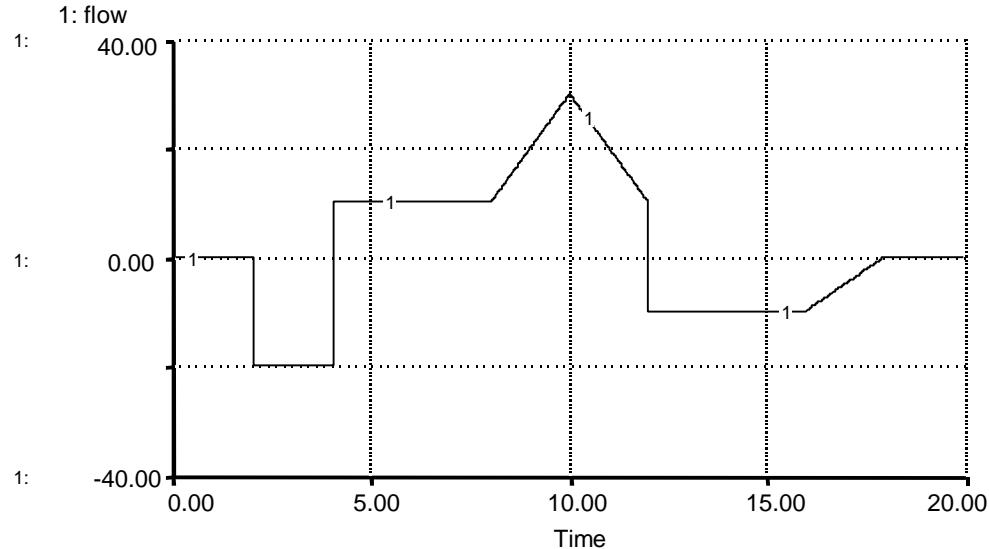
3. Graphical Integration Exercises Part 4: Reverse Graphical Integration

Please read this paper and do all the included exercises.

Using the skills that you acquired in the paper, please reverse integrate the following stock behavior:



From time = 0 to time = 2, the stock remains at 0. From time = 2 to time = 4, the stock decreases linearly from 0 to -40. From time = 4 to time = 8, the stock increases linearly from -40 back to 0. From time = 8 to time = 10, the stock increases parabolically from 0 to 40. From time = 10 to time = 12, the stock increases by "negative parabolic" growth from 40 to 80. From time = 12 to time = 16, the stock decreases linearly from 80 to 40. From time = 16 to time = 18, the stock decreases parabolically from 40 to 30. The stock then remains at 30 until time = 20.



The stock is constant from time = 0 to time = 2, so the flow is equal to 0.

From time = 2 to time = 4, the stock is decreasing linearly with a slope of -20 , so the flow value is -20 .

As the stock increases linearly from -40 to 0 from time = 4 to time = 8, the flow is constant at $+10$.

From time = 8 to time = 10, the stock increases parabolically from 0 to 40 , so the flow is increasing linearly from $+10$ to $+30$, with slope of $+10$.

From time = 10 to time = 12, the stock increases by “negative parabolic” growth from 40 to 80 , so the flow is still positive but decreasing linearly from $+30$ back to $+10$, with slope of -10 .

From time = 12 to time = 16, the stock decreases linearly from 80 to 40 , so the flow is constant at -10 .

From time = 16 to time = 18, the stock decreases parabolically from 40 to 30 , so the flow is negative but increases linearly from -10 to 0 , with slope of $+5$.

From time = 18 to time = 20, the stock is constant, so the flow equals 0.

4. Independent Modeling Exercise: Internet

In Assignment 1, you were asked to talk about the forces behind the explosive growth of the Internet, as well as the limits to its growth. In this exercise, we will be analyzing some of the simple systems you came up with.

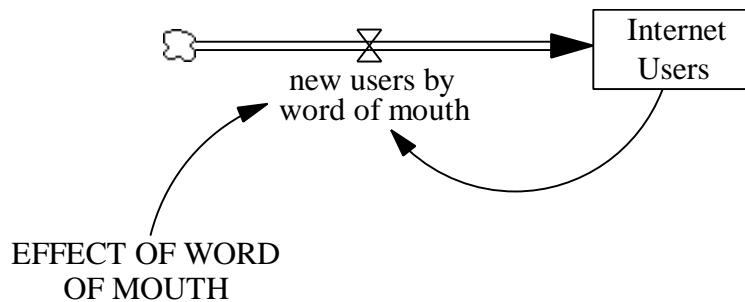
A. Perhaps the strongest force attracting people to the Internet is word-of-mouth. One person surfs, loves it, and tells his friends about it. His friends then decide to try it out also, then in turn tell their friends, and so on. Let's assume that each user influences 0.5 friends per year, and that there are initially 1 million users of the internet in the US at the time of our study (we will only be concerned with the United States in this problem).

Identify the elements of the system, label each as either a stock, flow, or constant, and label its units.

Simulate the model for 30 years and examine the behavior (remember to use an appropriate time step). In your assignment solutions document, include the model diagram, documented equations, and a graph of model behavior.

internet users	stock (people)
new users by word of mouth	flow (people/year)
effect of word of mouth	constant (1/year)

Model diagram:



Model equations:

$$\text{EFFECT OF WORD OF MOUTH} = 0.5$$

Units: 1/Year

Fraction of people per year who become convinced to be an internet user.

$$\text{Internet Users} = \text{INTEG}(\text{new users by word of mouth}, 1e+006)$$

Units: people

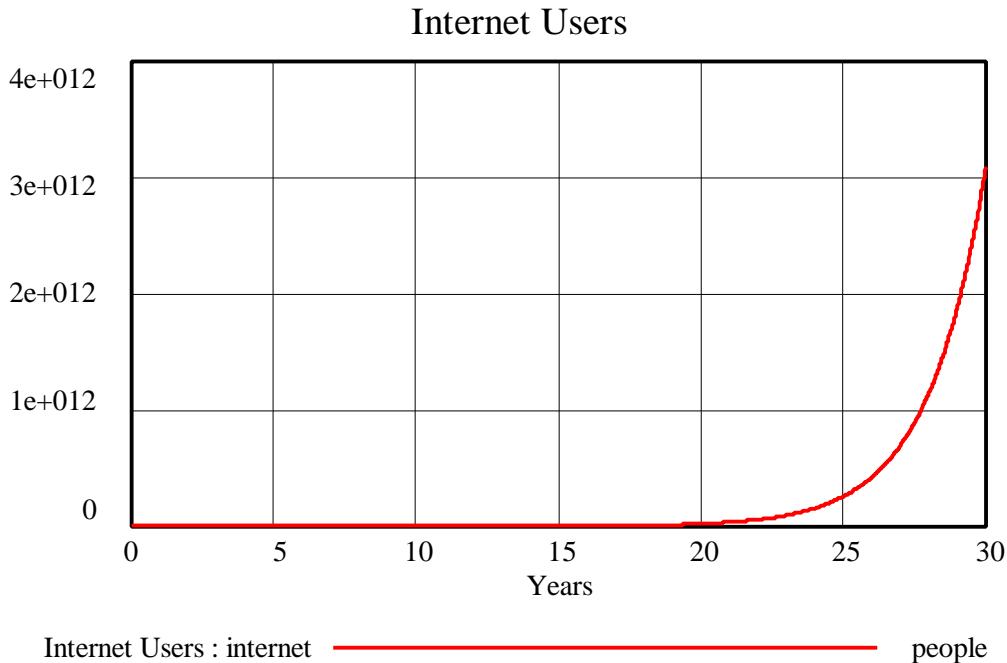
Number of people who are already internet users.

$$\text{new users by word of mouth} = \text{Internet Users} * \text{EFFECT OF WORD OF MOUTH}$$

Units: people/Year

The number of people per year who become internet users.

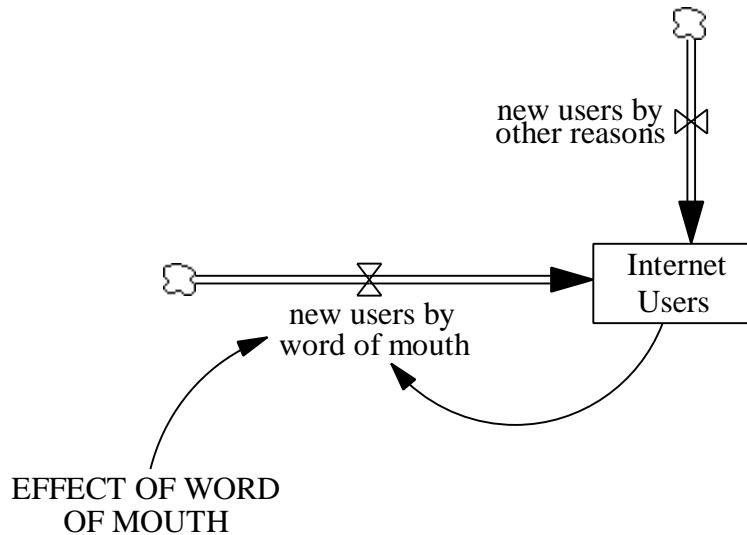
Model behavior:



As expected, the model generates exponential growth, with doubling time of 0.7 / 0.5 per year = 1.4 years. The stock reaches unreasonably high values because the model does not yet place any limit on its growth.

B. In Part A, we did not account for the fact that as colleges and workplaces offer access to the Internet, a somewhat constant number of people is being exposed to the Internet annually, making them internet users. Let's say that an additional 100,000 people become internet users every year simply due to circumstance. Add this flow to the model from part A and simulate the model for 30 years. In your assignment solutions document, include the new model diagram, modified equations, and a graph of model behavior.

Model diagram:



Modified model equations:

Internet Users = INTEG (new users by word of mouth + new users by other reasons,
 $1e+006$)

Units: people

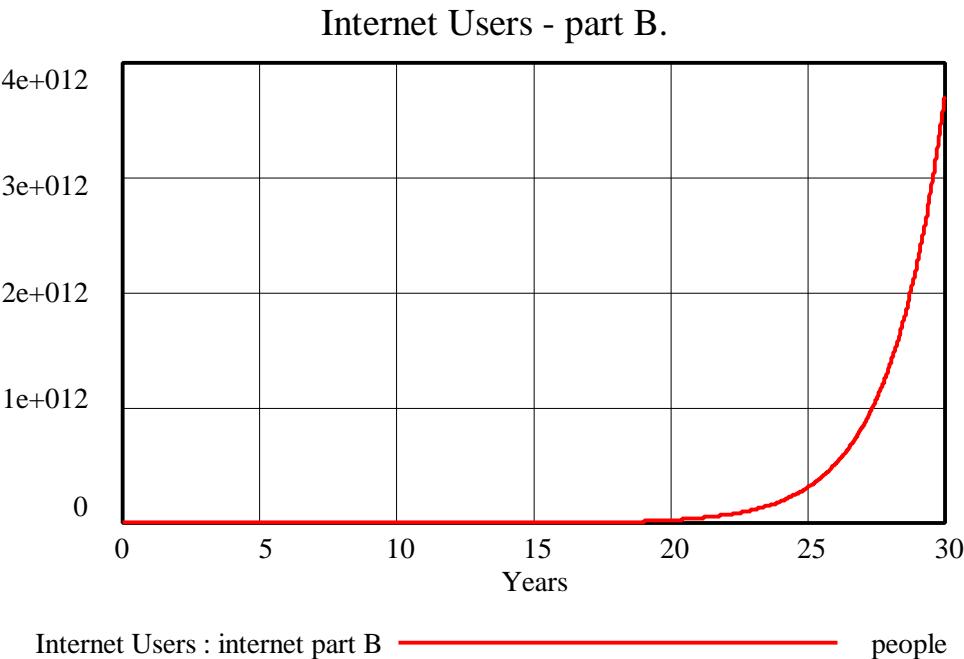
Number of people who are already internet users.

new users by other reasons = 100000

Units: people/Year

The number of people per year who become internet users simply due to circumstance.

Model behavior:



System behavior is still exponential, but the stock reaches even higher values. The constant inflow of “new users by other reasons” not only adds to the stock of internet users directly but also compounds the positive-feedback behavior. Therefore, a small addition of 100,000 per year leads to big gains over time.

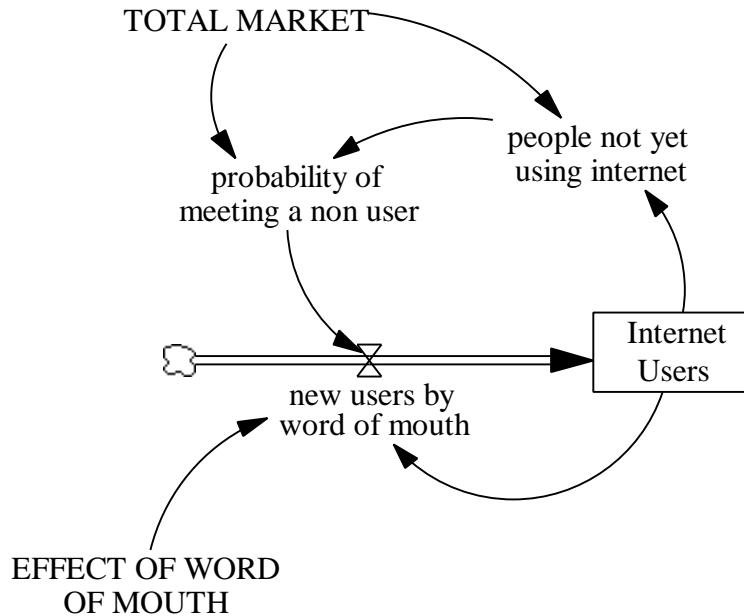
C. Often, hardware limitations prevent some that wish to get on the Internet from doing so. The growth of Internet service providers (ISPs) has not come close to the growth of demand for Internet access. What if the total number of people who have the hardware capability to be an Internet user is limited to 15 million? (Each user still influences 0.5 friends per year). In this scenario, you do not need to account for the inflow of 100,000 people from part B. Modify the model from part A and simulate the model for 30 years. In your assignment solutions document, include the new model, documented equations, and graphs of model behavior. Explain the behavior that you observe.

Note: Keep in mind that special functions, such as IF-THEN-ELSE, should not be used unless absolutely appropriate-and it is not appropriate in this situation.

In order to capture the limited market size, the model should contain a negative-feedback loop. While the persuasive abilities of “Internet Users” remain the same over time (the “EFFECT OF WORD OF MOUTH” is constant), as the number of “Internet Users” grows, it becomes more difficult for them to meet someone who is not yet a user but has the hardware capabilities to become a user. Hence, the model should include a variable called “probability of meeting a non user,” which is just the ratio of “people not yet on internet” to the “TOTAL MARKET.”

The new model is as follows:

Model diagram:



Model equations:

$$\text{EFFECT OF WORD OF MOUTH} = 0.5$$

Units: 1/Year

Fraction of people per year who become convinced to be an internet user.

$$\text{Internet Users} = \text{INTEG}(\text{new users by word of mouth}, 1e+006)$$

Units: people

Number of people who are already internet users.

$$\text{new users by word of mouth} = \text{Internet Users} * \text{EFFECT OF WORD OF MOUTH} * \text{probability of meeting a non user}$$

Units: people/Year

The number of people per year who become internet users.

$$\text{people not yet using internet} = \text{TOTAL MARKET} - \text{Internet Users}$$

Units: people

The number of people capable of becoming an internet users who are not users yet.

$$\text{probability of meeting a non user} = \text{people not yet using internet} / \text{TOTAL MARKET}$$

Units: dmnl

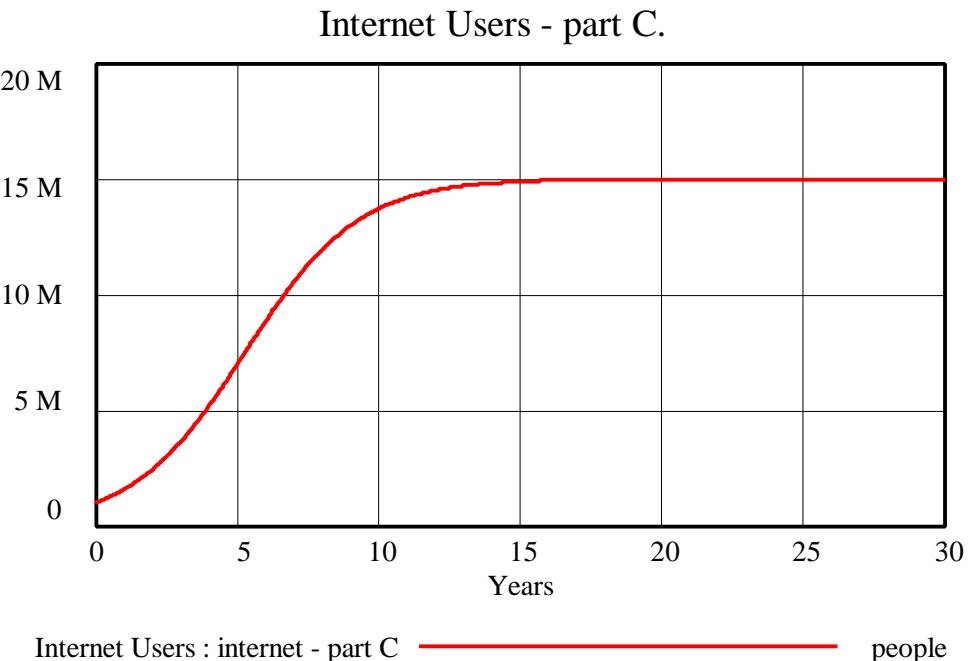
The percentage of people in the total market who have not yet become internet users.

$$\text{TOTAL MARKET} = 1.5e+007$$

Units: people

The total number of people with the hardware capabilities to become internet users.

You should not model the system as a simple goal-gap negative-feedback loop. A simple negative-feedback loop formulation does not realistically capture the initial spread of the internet. In reality, the internet grows slowly at first, when the number of users is small, and the growth becomes faster and faster as the number of users increases, due to word of mouth. With a goal-gap negative-feedback loop structure, however, the system tries to close the gap between the “Internet Users” and the “TOTAL MARKET” quickly, and then the growth slows down as the gap becomes smaller. The model should therefore contain both a positive and a negative feedback loop, and should generate a shift in loop dominance from the positive to the negative loop.



The system now exhibits S-shaped growth. Initially, when the number of “Internet Users” is small, the size of the “TOTAL MARKET” does not yet limit the growth, so the stock grows exponentially. As more and more people from the “TOTAL MARKET” become “Internet Users,” however, the “probability of meeting a non user” declines, and so does the flow of “new users by word of mouth.” The growth of “Internet Users” slows down and the stock asymptotically approaches its goal value.

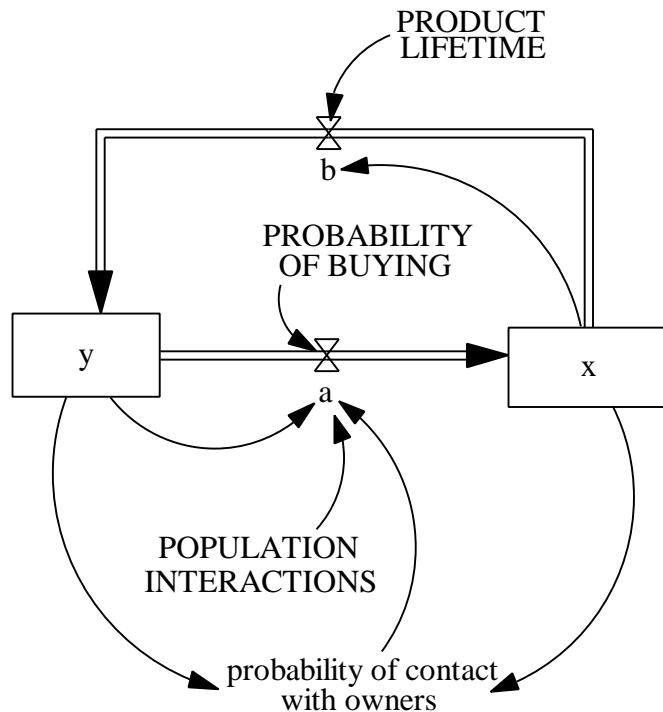
Appendix:

Mathematical derivation of the transitioning point between S-shaped and asymptotic behavior.

For ease of calculations, replace the variable names with the following variables:

Value of Owners = x
 Value of Non-Owners = y
 Value of buying = a
 Value of product break rate = b

The model now looks like this:



Remember we had said that the net flow for Owners is buying - product break rate.

Letting the net flow $\frac{dx}{dt} = \dot{x}$, the equation reads:

$$\dot{x} = a - b = 16 * y * 0.25 * \frac{x}{2000} - \frac{x}{2}$$

$$\dot{x} = 4 * y * \frac{x}{2000} - \frac{x}{2}$$

$$\text{and } y = 2000 - x$$

Using the two latter equations, we want to solve for a range of x when its net flow is growing. This means that the “buying” rate (a) is increasing faster than the “product break rate” (b). In mathematical notation, the inequality is:

$$\dot{a} > \dot{b}, \text{ or } \dot{a} - \dot{b} > 0$$

We can then simply rewrite the inequality in terms of x by:

$$\dot{x} = a - b = -\dot{y}$$

$$\ddot{x} = \dot{a} - \dot{b}$$

$$\ddot{x} > 0$$

For the system to exhibit the exponential stage of S-shaped behavior, the second derivative of x must be greater than 0. By taking derivatives and using the product rule, we get that the change in \dot{x} , or \ddot{x} , is:

$$\ddot{x} = \frac{4}{2000} * (y * \dot{x} + x * \dot{y}) - \frac{\dot{x}}{2}$$

Substitute for y and the derivative of y:

$$\ddot{x} = \frac{4}{2000} * ((2000 - x) * \dot{x} + x * (-\dot{x})) - \frac{\dot{x}}{2} > 0$$

$$\ddot{x} = \frac{4}{2000} * (2000 * \dot{x} - 2 * x * \dot{x}) - \frac{\dot{x}}{2} > 0$$

$$\ddot{x} = \frac{4 * \dot{x}}{2000} * (2000 - 2 * x) - \frac{\dot{x}}{2} > 0$$

$$\ddot{x} = \dot{x} \left(4 - \frac{4 * x}{1000} - \frac{1}{2} \right) > 0$$

$$\ddot{x} = 3.5 - \frac{4 * x}{1000} > 0$$

$$3500 > 4x$$

$$875 > x$$

We have just proven that when the initial number of “Owners” is less than 875, the rate of “buyers” increases faster than the product break rate, so the net flow into “Owners” increases in magnitude, causing the exponential stage of S-shaped growth. The value 875 is the “N” value we were searching for. When the simulation starts out with an initial value of “Owners” greater than 875, the net flow into “Owners” can only decrease in magnitude, leading to asymptotic convergence to equilibrium value of 1750. Similarly, the system can only exhibit S-shaped behavior when “Non Owners” is greater than 2000 - 875, or “Non Owners” > 1125.