

**Guided Study Program in System Dynamics**  
System Dynamics in Education Project  
System Dynamics Group  
MIT Sloan School of Management<sup>1</sup>

Solutions to Assignment #10  
December 9, 1998

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***Reading Assignment:***

*Please refer to Road Maps 3: A Guide to Learning System Dynamics (D-4503-4) and read the following:*

- *Study Notes in System Dynamics,<sup>2</sup> by Michael Goodman, Chapters 4 and 5*

*Then refer to Road Maps 5: A Guide to Learning System Dynamics (D-4505-4) and read the following paper from Road Maps 5:*

- *Beginner Modeling Exercises Section 5: Mental simulation of Combining Feedbacks in First-Order Systems, by Laughton Stanley and Helen Zhu (D-4593-1)*

*Please download and read the following paper from <http://sysdyn.mit.edu/gsp98/>*

- *Mistakes and Misunderstandings: Time Constants and Decay Fractions, by Mila Getmansky (D-4679)*

***Exercises:***

1. *Study Notes in System Dynamics, Chapter 4: S-Shaped Growth Structure*

*Please note that even though the equations of the “S-shaped growth structure” shown in Figure 4-3 of the Study Notes generate S-shaped growth, the structure was not designed to represent any real-world system. When modeling a system that exhibits S-shaped growth, such as the rat population system in Figure 4-10, you could make a scatter plot of the level values vs. the rate values, and you would obtain a graph similar to the “Rate value” variable in Figure 4-3, or to the graphs shown in Figure 4-6. In fact, Figure 4-12 shows such a scatter plot. In general, we will not use level vs. rate plots but instead will concentrate on graphs depicting the level and the rate behaviors over time.*

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<sup>2</sup> Goodman, Michael R., 1974. *Study Notes in System Dynamics*, Waltham, MA: Pegasus Communications. 388 pp.

A. Please build the rat population model shown in Figure 4-10. Make sure that the table function for “infant survival multiplier” has a dimensionless input. To do this, you will need to define several additional variables, not shown in Figure 4-10.

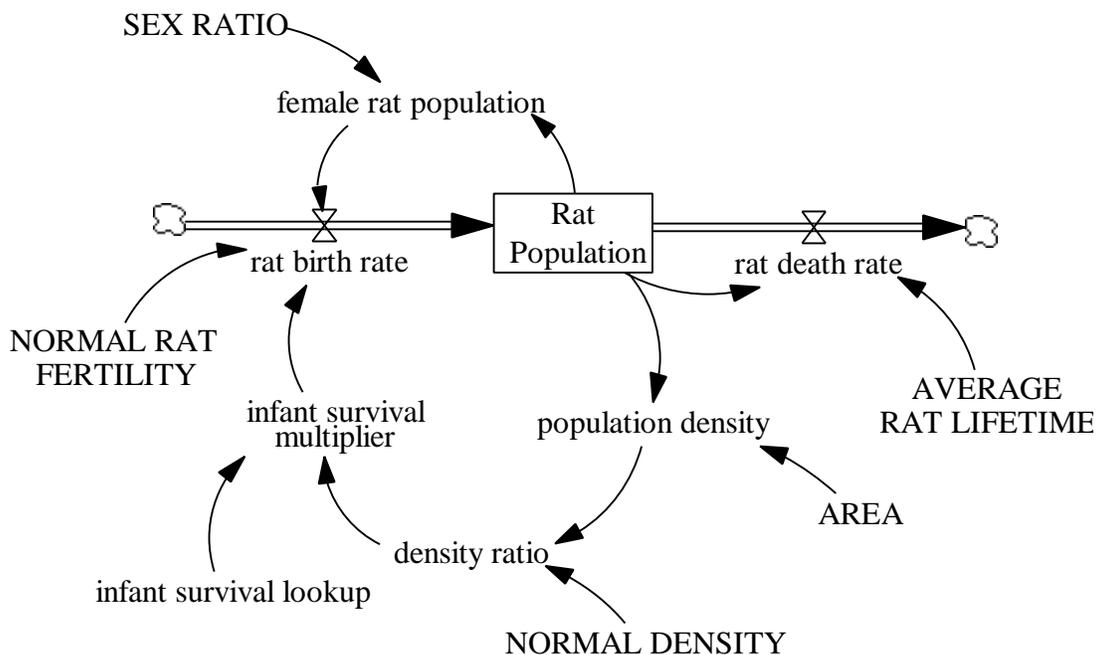
- the “NORMAL RAT DENSITY” equals 0.0025 rats per squared foot.
- the “density ratio” is the ratio of “population density” to the “NORMAL RAT DENSITY”
- the “infant survival lookup” is a table function that looks just like the table function in Figure 4-11, but the scale of the input, the “density ratio”, ranges from 0 to 10
- the “infant survival multiplier” is the “infant survival lookup” function of the “density ratio”:  $\text{infant survival multiplier} = \text{infant survival lookup}(\text{density ratio})$
- the “NORMAL RAT FERTILITY” equals 0.4 rats per female per month

Notice that the “NORMAL RAT DENSITY” was chosen somewhat arbitrarily to equal 0.0025 rats per square foot. From the table function in Figure 4-11, one can see that the “infant survival multiplier” is equal to 1 for values of the “population density” less than and equal to 0.0025 rats per square foot. Hence, up to this density, the “rat birth rate” will equal the “NORMAL RAT FERTILITY” multiplied by the “female rat population.”

With the above equations, your model should be dimensionally consistent, and your lookup function will have normalized inputs and outputs.

In your assignment solutions document, please include the model diagram, a graph of the lookup function, documented equations, and a graph of model behavior.

Model diagram:



Model equations:

$$\text{AREA} = 11000$$

Units: squared feet

The area of the confined space in which the rats live.

$$\text{AVERAGE RAT LIFETIME} = 22$$

Units: Month

The average lifetime of a rat.

$$\text{density ratio} = \text{population density} / \text{NORMAL DENSITY}$$

Units: dmnl

The ratio of the current population density to the normal rat density.

$$\text{female rat population} = \text{Rat Population} * \text{SEX RATIO}$$

Units: female

The number of female rats in the total rat population.

$$\text{infant survival lookup} ([ (0,0) - (10,1) ], (0,1), (1,1), (2,0.96), (3,0.92), (4,0.82), (5,0.7), (6,0.52), (7,0.34), (8,0.2), (9,0.14), (10,0.1))$$

Units: dmnl

The lookup function for the infant survival multiplier.

$$\text{infant survival multiplier} = \text{infant survival lookup}(\text{density ratio})$$

Units: dmnl

The infant survival multiplier is a function of the density ratio.

$$\text{NORMAL DENSITY} = 0.0025$$

Units: rats/squared feet

The number of rats per squared foot at the normal conditions, when the rat fertility equals the normal rat fertility.

$$\text{NORMAL RAT FERTILITY} = 0.4$$

Units: (rats/female)/Month

The normal number of rats born per female rat every month.

$$\text{population density} = \text{Rat Population} / \text{AREA}$$

Units: rats/squared feet

The number of rats per squared foot.

$$\text{rat birth rate} = \text{female rat population} * \text{NORMAL RAT FERTILITY} * \text{infant survival multiplier}$$

Units: rats/Month

The number of rats born every month.

$$\text{rat death rate} = \text{Rat Population} / \text{AVERAGE RAT LIFETIME}$$

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Units: rats/Month

The number of rats dying every month.

Rat Population = INTEG (rat birth rate - rat death rate, 10)

Units: rats

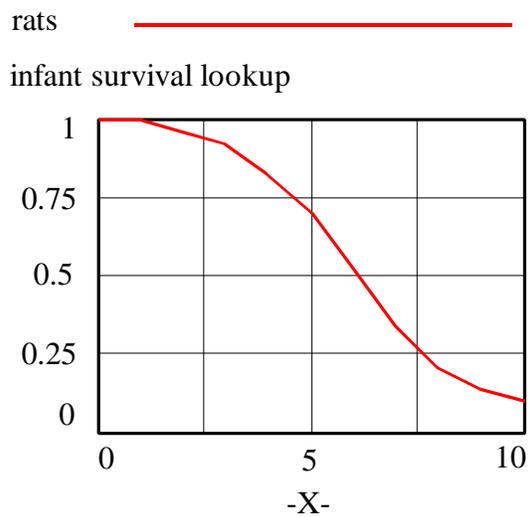
The number of rats in the population.

SEX RATIO = 0.5

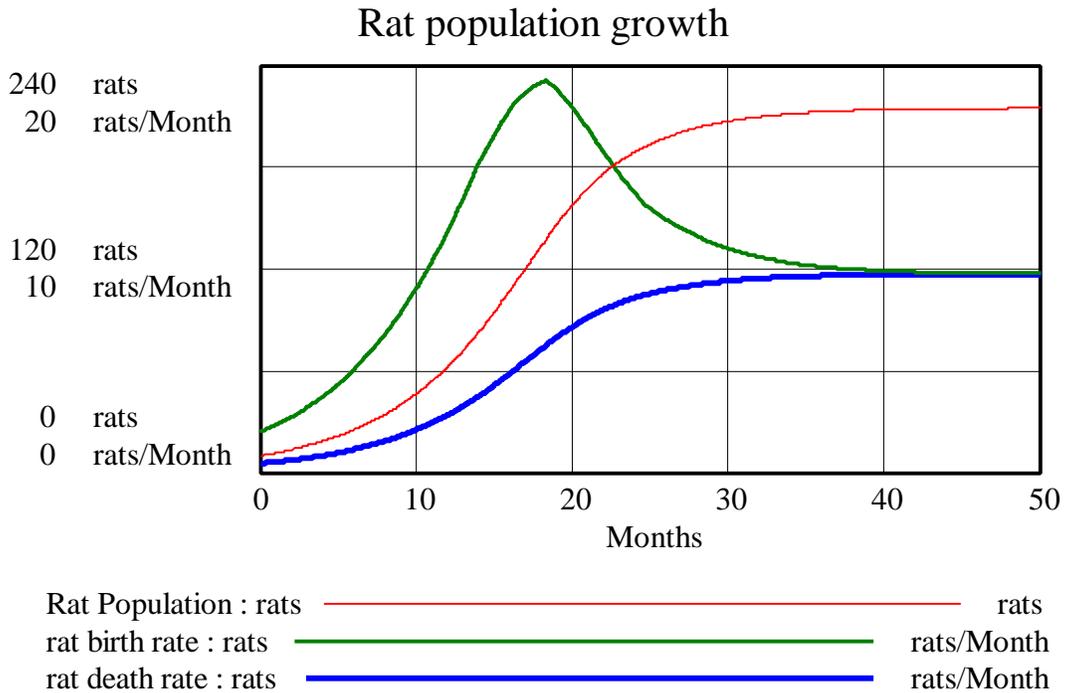
Units: female/rats

The ratio of female rats to the total rat population.

Graph of the lookup function:

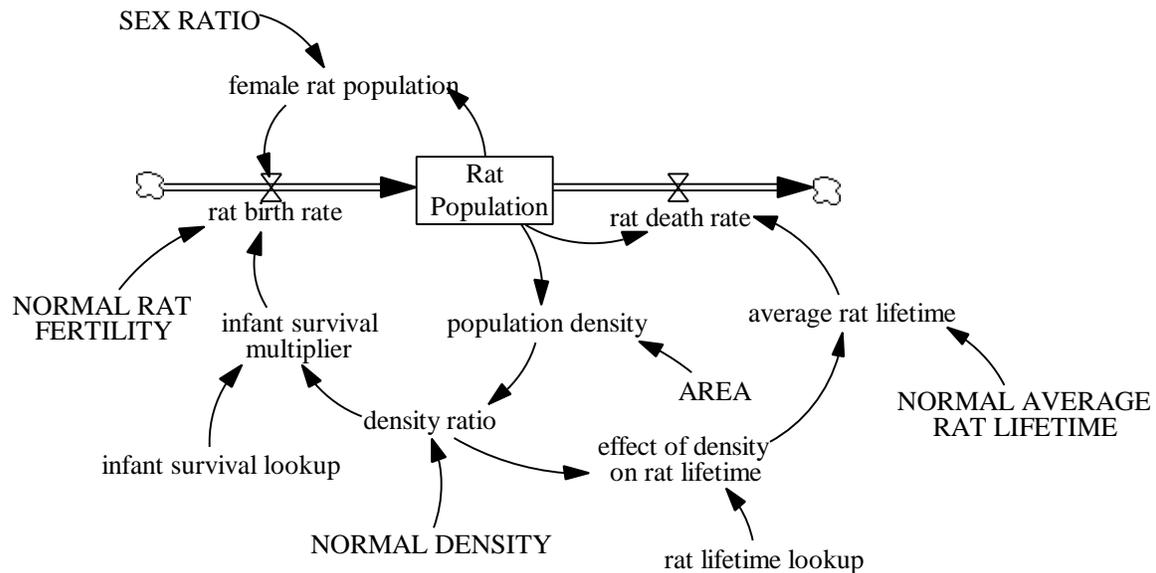


Model behavior:



*B. As mentioned in the Study Notes, this model accounts for only one density effect. The “rat death rate” is also likely to be increased because of a higher density of rats. Modify the model built in part A to include another table function, the “effect of density on rat lifetime.” In your assignment solutions document, please include the new model diagram, documented equations, and a graph of the new lookup function.*

Model diagram:



Model equations:

$$\text{AREA} = 11000$$

Units: squared feet

The area of the confined space in which the rats live.

$$\text{average rat lifetime} = \text{NORMAL AVERAGE RAT LIFETIME} * \text{effect of density on rat lifetime}$$

Units: Month

The actual average rat lifetime changes as the density changes.

$$\text{density ratio} = \text{population density} / \text{NORMAL DENSITY}$$

Units: dmnl

The ratio of the current population density to the normal rat density.

$$\text{effect of density on rat lifetime} = \text{rat lifetime lookup}(\text{density ratio})$$

Units: dmnl

The rat lifetime multiplier is the rat lifetime lookup function of the density ratio.

$$\text{female rat population} = \text{Rat Population} * \text{SEX RATIO}$$

Units: female

The number of female rats in the total rat population.

$$\text{infant survival lookup}([(0,0) - (10,1)], (0,1), (1,1), (2,0.96), (3,0.92), (4,0.82), (5,0.7), (6,0.52), (7,0.34), (8,0.2), (9,0.14), (10,0.1))$$

Units: dmnl

The lookup function for the infant survival multiplier.

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infant survival multiplier = infant survival lookup(density ratio)

Units: dmnl

The infant survival multiplier is a function of the density ratio.

NORMAL AVERAGE RAT LIFETIME = 22

Units: Month

The average lifetime of a rat.

NORMAL DENSITY = 0.0025

Units: rats/squared feet

The number of rats per squared foot at the normal conditions, when the rat fertility equals the normal rat fertility.

NORMAL RAT FERTILITY = 0.4

Units: (rats/female)/Month

The number of rats born per female rat every month.

population density = Rat Population / AREA

Units: rats/squared feet

The number of rats per squared foot.

rat birth rate = female rat population \* NORMAL RAT FERTILITY \* infant survival multiplier

Units: rats/Month

The number of rats born every month.

rat death rate = Rat Population / average rat lifetime

Units: rats/Month

The number of rats dying every month.

rat lifetime lookup([(0,0)-(10,1)], (0,1), (1,1), (2,0.98), (3,0.95), (4,0.9), (5,0.84), (6,0.76), (7,0.65), (8,0.5), (9,0.3), (10,0.05))

Units: dmnl

The lookup function for the rat lifetime.

Rat Population = INTEG (rat birth rate - rat death rate, 10)

Units: rats

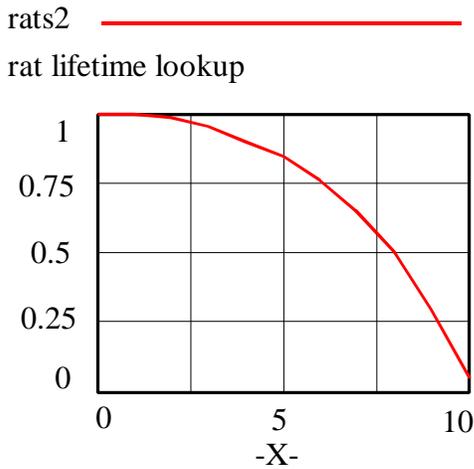
The number of rats in the population.

SEX RATIO = 0.5

Units: female/rats

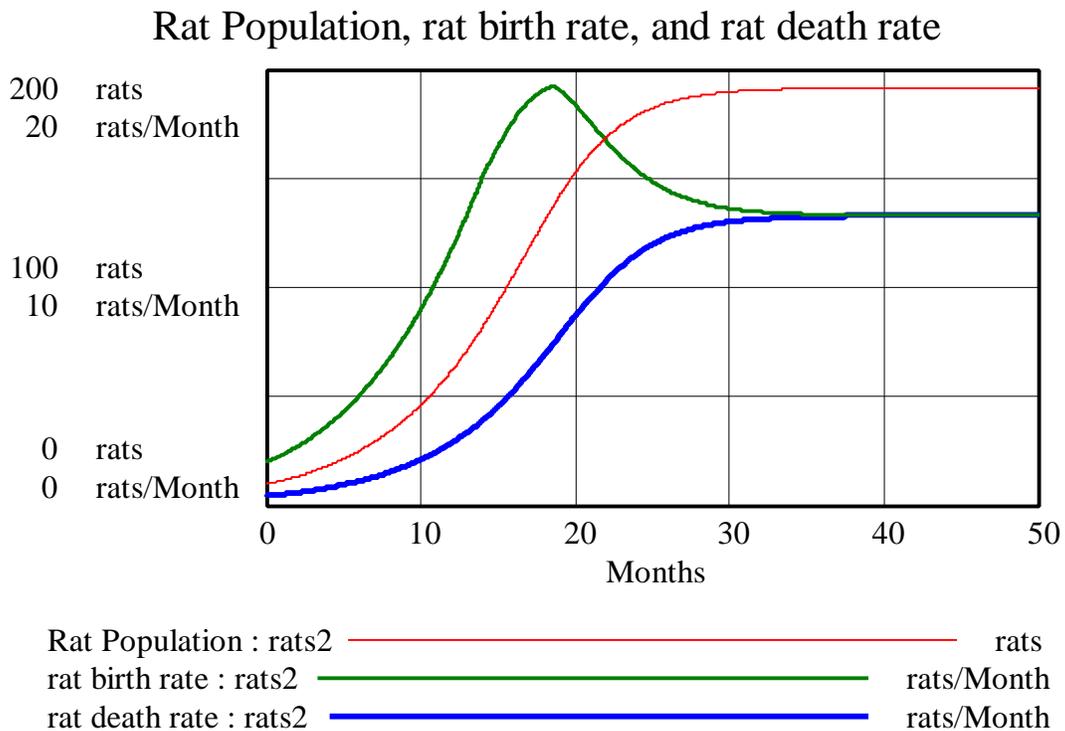
The ratio of female rats to the total rat population.

Graph of the “rat lifetime lookup” function:



*C. Before simulating the model, draw a reference mode for the stock “Rat Population.” Then simulate the model. Does the model generate the behavior you expected? Why or why not? Explain any differences in behavior between this model and the original model from part A. In your assignment solutions document, include graphs of the behavior of the stock and the two flows.*

Model behavior:

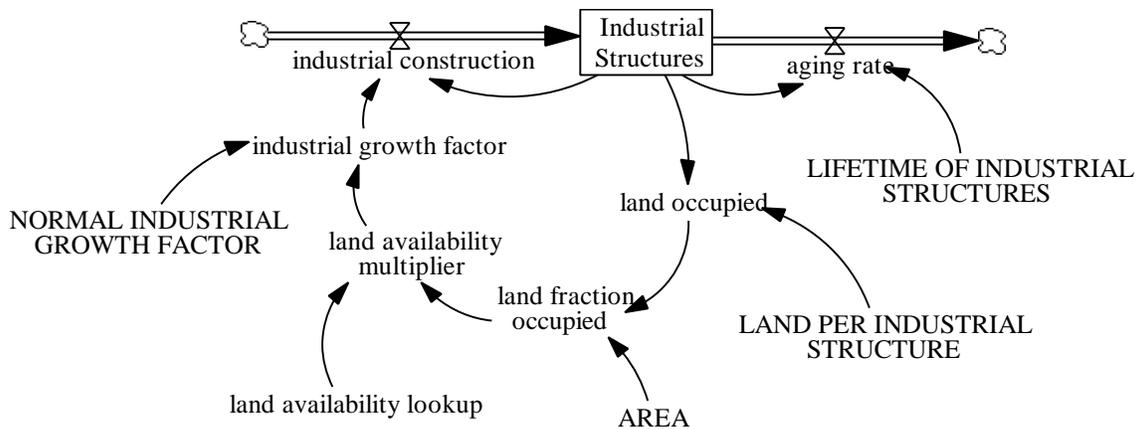


Compared to the simulation from part A, the “Rat Population” stabilizes at a lower equilibrium value because of the additional negative-feedback loop that lowers the rat lifetime as the population rises. As the “Rat Population” increases, the “average rat lifetime” falls, so the “rat death rate” and hence the “rat birth rate” are both higher in equilibrium than in the simulation from part A.

2. Study Notes in System Dynamics, Chapter 5: Review of Simple Structures: Industrial Land-Use Model

Follow the chapter and build the complete industrial land-use model as shown in Figure 5-20. In your assignment solutions document, include the model diagram, documented equations, a graph of the “land availability multiplier” table function, and a graph of model behavior. Remember to modify the model from Figure 5-20 so that it contains all the variables required to formulate a lookup function in Vensim PLE, as well as all the “normal” values of variables.

Model diagram:



Model equations:

aging rate = Industrial Structures / LIFETIME OF INDUSTRIAL STRUCTURES  
 Units: structure/Year  
 The number of structures that are demolished because of aging.

AREA = 100  
 Units: acres  
 The total land area.

industrial construction = Industrial Structures \* industrial growth factor  
 Units: structure/Year  
 The number of industrial structures built every year.

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industrial growth factor = NORMAL INDUSTRIAL GROWTH FACTOR \* land availability multiplier

Units: 1/Year

The industrial growth factor changes as the land fraction occupied changes.

Industrial Structures = INTEG (industrial construction - aging rate, 5)

Units: structure

The number of industrial structures.

land availability lookup ((0,0) - (1,1]), (0,1), (0.1,1), (0.2,0.98), (0.3,0.95), (0.4,0.75), (0.5,0.55), (0.6,0.35), (0.7,0.2), (0.8,0.1), (0.9,0.05), (1,0))

Units: dmnl

The lookup function for the land availability multiplier.

land availability multiplier = land availability lookup(land fraction occupied)

Units: dmnl

The land availability multiplier is the land availability lookup function of the land fraction occupied.

land fraction occupied = land occupied / AREA

Units: dmnl

The land fraction occupied is the fraction of the total area that is already occupied by industrial structures.

land occupied = Industrial Structures \* LAND PER INDUSTRIAL STRUCTURE

Units: acres

The amount of land already occupied by industrial structures.

LAND PER INDUSTRIAL STRUCTURE = 1

Units: acres/structure

The amount of land required per industrial structure.

LIFETIME OF INDUSTRIAL STRUCTURES = 20

Units: Year

The average lifetime of an industrial structure.

NORMAL INDUSTRIAL GROWTH FACTOR = 0.1

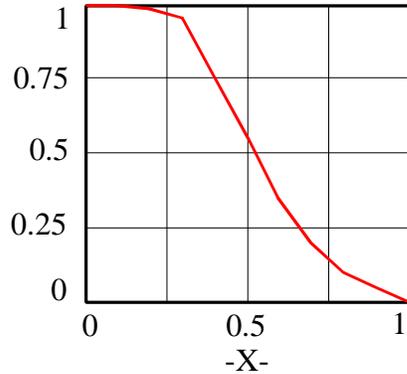
Units: 1/Year

The industrial growth factor under normal conditions of the system.

Graph of the “land availability lookup” function:

land —————

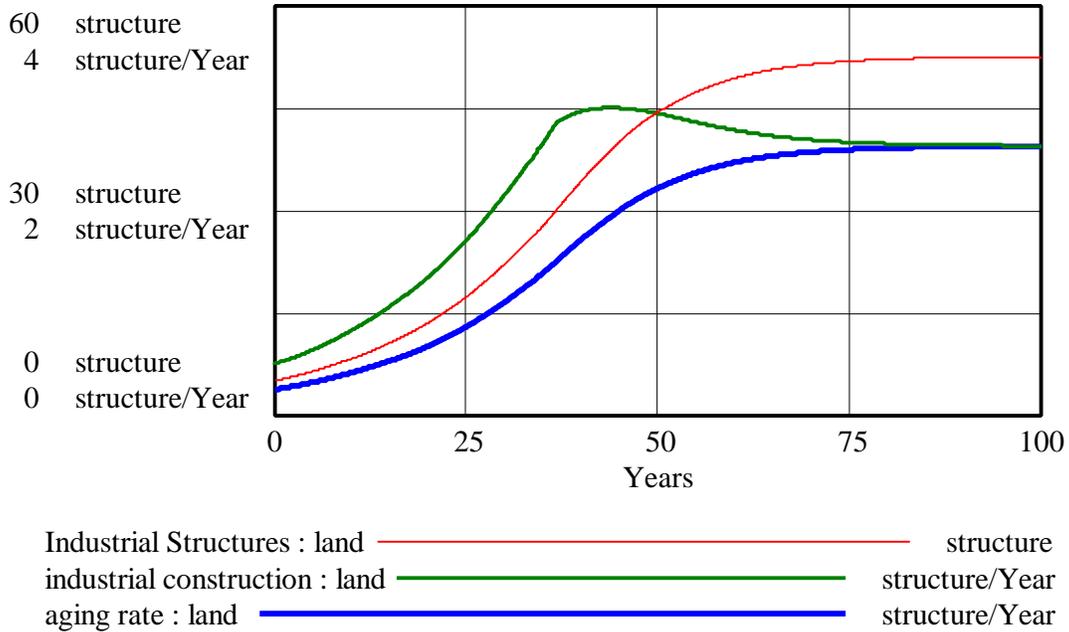
land availability lookup



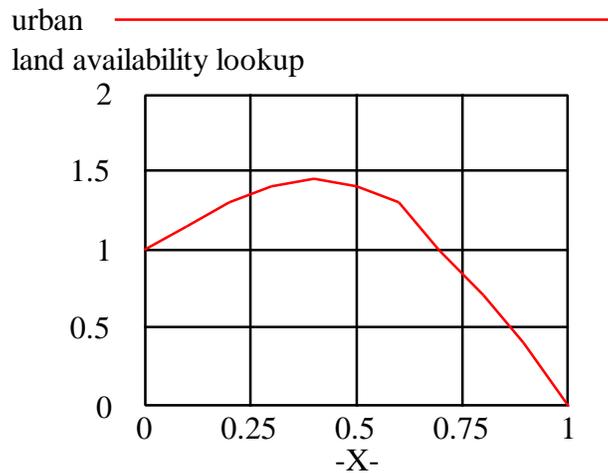
Some of the participants modeled the system using a “NORMAL LAND FRACTION OCCUPIED,” and then used the ratio of “land fraction occupied” to the “NORMAL LAND FRACTION OCCUPIED” as the input to the lookup function for “land availability multiplier.” Both formulations are technically and dimensionally correct; however, the formulation with “NORMAL LAND FRACTION OCCUPIED” might be less intuitively understandable. A normal value of a variable is usually inserted to obtain a dimensionless value as the multiplier input, but that is not necessary in this situation because “land fraction occupied” is already a dimensionless fraction. If the formulation with “NORMAL LAND FRACTION OCCUPIED” is used, the normal value should be chosen to have some ‘normal’ meaning in the real life system; for example, a “NORMAL LAND FRACTION OCCUPIED” equal to 0.1 seems rather low for a ‘normal’ value.

Model behavior:

Industrial Structures, industrial construction, and aging rate



The “land availability lookup” function can also be formulated slightly differently, as is done in Appendix A of *Urban Dynamics*, page 174, Figure A-32. The table function used in *Urban Dynamics* is:



The shape of the above table function is explained on pages 173-4 of *Urban Dynamics* as follows:

“At very low levels of land occupancy, the area has not yet shown economic importance and the nearby empty land tends to discourage aggressive construction. As land occupancy increases towards the middle region, the rising multiplier indicates economic

excitement and potential. Also, all of the common urban activities and services are now present to support further expansion. But as the area approaches the fully occupied condition, the more favorable construction sites have already been used. The less and less attractive locations begin to depress the tendency to build until the absolute unavailability of land drives the construction rate to 0 at full land occupancy.”

3. Beginner Modeling Exercises Section 5: Mental simulation of Combining Feedbacks in First-Order Systems

*After reading this paper and doing all the included exercises, answer the following question:*

*This paper presented a first-order system that produces S-shaped behavior, where initial exponential growth is hindered by constraints on the system. Think of a system you are familiar with that exhibits S-shaped growth. Explain clearly the positive and negative feedback loops that are driving the system behavior and why you believe there will be a shift in loop dominance.*

The growth in the number of users of new products, such as the Internet or mobile telephones, are examples of a system producing S-shaped behaviors.

At first, the number of users shows exponential growth, typical of a positive-feedback loop: every user of the Internet (or a mobile telephone) spreads the news about the wonders of the new product, which attracts new users. The more users there are already, the more new users are attracted.

As the number of users increases, however, the number of potential users decreases, because the population is finite. As the number of potential users decreases, each user is able to attract fewer new users. The typical behavior of a negative-feedback loop appears.

As both loops interact, growth goes from exponential to asymptotic and then to stable equilibrium when all the potential users have become users.

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One system I can think of is my daily accomplishments at work. In the morning, I start out rested and motivated. As the day progresses, I accomplish tasks, which helps to further motivate me to accomplish more and more tasks. However, for each task I accomplish I cause some fatigue. The longer I work the more fatigue I generate. Finally, towards the end of the day, the fatigue wears me out totally, all my motivation is gone, and I can accomplish nothing else. Two loops are in action here. The first is the positive-feedback loop of motivation and tasks accomplished, and the second is the negative-feedback loop that erodes my motivation as more tasks are accomplished. The shift in dominance occurs when my motivation peaks. My motivation curve is bell shaped. It starts out low in the early morning and soars up during the day but eventually stops rising and begins descending towards zero. Because motivation is what drives the number of tasks accomplished this causes an S-shaped curve of my Completed Tasks for the day.

Numbers of individuals in a new flock of gregarious birds may exhibit s-shaped behavior. At first the squawking and flapping of the flock attracts new members rapidly and in proportion to its size. Gradually pecking, bickering, and other stressful bird brained behavior causes birds to leave the flock at an increasing rate as the flock grows. This relation can be described with an “effect of flock size on leaving” lookup function growing from 1 at and below normal conditions to 6 at flock sizes 10 times normal or larger. In this system, the positive-feedback loop affecting the number of birds joining the flock is dominant when the flock size is small and causes initial growth at an ever-increasing rate. As flock size grows, the effect of flock size on leaving grows rapidly, and this negative-feedback loop becomes dominant. The dominance of this negative feedback loop causes the system to approach equilibrium.

#### 4. Mistakes and Misunderstandings: Time Constants and Decay Fractions

*Read this paper carefully. You do not have to answer any questions for this paper but if you can think of an instance when you made the same mistake, feel free to share with us the lesson you learned.*

I am glad that you had us read this paper. I now understand better the concept of “units per units per time.” Before this, the concept of “1/time” was a bit abstract. The example of eating half of the peanuts *over a period of time* helped clarify this.