Second-Order Systems

Prepared for the
MIT System Dynamics in Education Project
Under the Supervision of
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1. Abstract

Second-order systems can exhibit a number of different behaviors. This paper analyzes the different behaviors that can be generated by a simple second-order model with one feedback loop connecting the stocks. When parameters in the model are both positive, the loop is a positive feedback loop, and the model will usually exhibit exponential growth or decline. When both parameters are negative, the loop is again a positive feedback loop, and the model will also tend to exhibit exponential growth or decline. When the two parameters are opposite in sign, however, the loop is a negative feedback loop, and the model will generate sustained oscillations.

2. Introduction

This paper examines different forms of behavior generated by simple second-order systems with one feedback loop connecting the stocks. The order of a system is a count of the number of stocks in the system. A second-order system is therefore a system with two stocks. Previous Road Maps papers[1] presented a simple second-order system as the generic structure for sustained oscillations. Simple second-order systems, however, can exhibit a range of different behaviors, as this paper demonstrates.

The models presented in this paper describe how two lovers interact. Different people have different personalities that dictate how they will respond to their partners' love. Some people love their partners more the more their partners love them. They get discouraged and give up if their partners do not love them. Other people, on the other hand, get annoyed if their lovers love them too much. They are attracted to people who are not attracted to them. This paper will match together lovers with different personalities and observe how their relationships unfold. Each pair will be represented by a different set of parameters in the model. By looking at the dynamic behavior generated by matching different personalities, that is, by simulating the model with different combinations of parameters, this paper will study the behavior of simple second-order systems.[2]

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[1] The two papers are: C. Chung, 1996, *Generic Structures in Oscillating Structures I* (D-4426-1); and K. Agastein, 1997, *Oscillating Systems 2: Sustained Oscillation* (D-4602), both published by the System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology.

[2] The idea of modeling love affairs, in particular between Romeo and Juliet, is taken from Strogatz, S.H., 1988, Love affairs and differential equations. *Math Magazine* 61, 35.

3. Romeo and Juliet

Romeo and Juliet[3] are madly in love with each other. With each secret meeting, Romeo's love for Juliet grows. Because he loves her, he does everything he can to impress her. Juliet is flattered by his attention and, in return, her love for Romeo also grows. Because Romeo senses that Juliet loves him, he allows his passion to soar. What will happen to Romeo and Juliet's love?

Figure 1 shows a model of Romeo and Juliet's love. In the model each stock represents the accumulation of one lover's love for another. The rate at which Romeo's love changes depends on the extent of Juliet's love and Romeo's personality. His personality is represented by a parameter that measures how he reacts to her love. Likewise, the change in Juliet's love equals the amount of Romeo's love times the extent to which she reacts to his love.

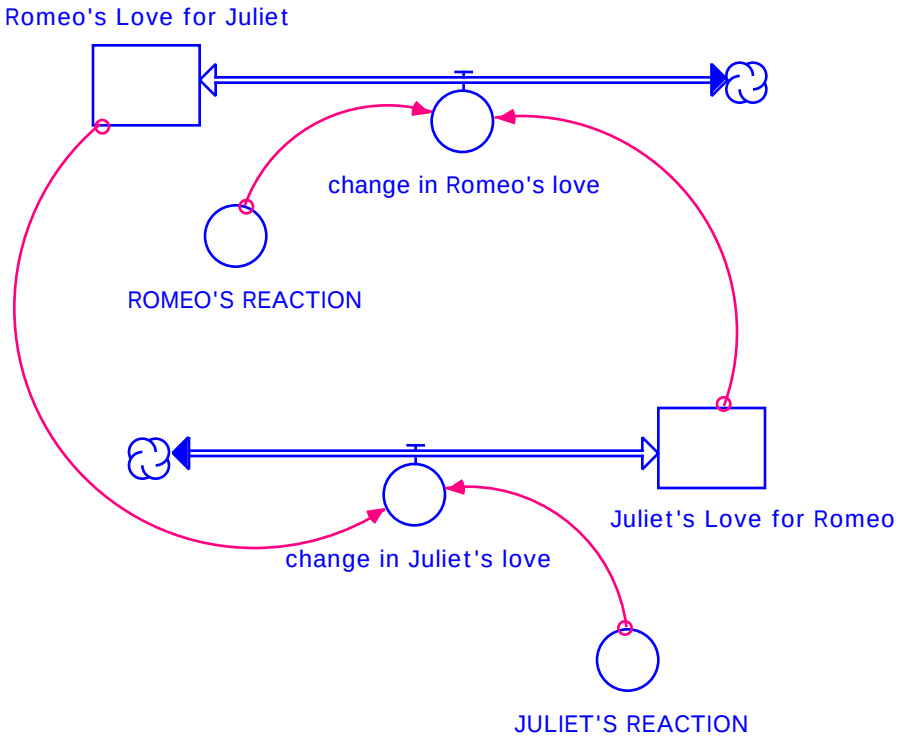

Figure 1: Model diagram for Romeo and Juliet

[3] from William Shakespeare, *Romeo and Juliet*.

The love that Romeo and Juliet feel for each other can be quantified in several ways. Love can be measured in terms of kisses per day, or tender thoughts per hour, for example. The units of the stocks of love in the models in this paper are based upon a scale of "love units." If the two protagonists are indifferent to each other, they feel zero love units. Positive love units represent how much they like each other, while negative love units represent how much they dislike each other. Right now, assume that initially Romeo and Juliet feel kindly disposed towards each other. They each feel one unit of love for the other.

The parameter "JULIET'S REACTION" represents how Juliet reacts to Romeo's love, that is, to what extent Romeo's love changes Juliet's love over time. Assume that every unit of Romeo's love increases Juliet's love by one unit each day. Therefore Juliet's reaction to Romeo's love is one love unit per love unit per day, or one per day. Romeo has the same personality as Juliet, so Romeo's reaction to Juliet's love is also one per day.

What will happen to Romeo and Juliet's love over the next twelve days? Mentally simulate the model. Simulating the model in STELLA generates the behavior reproduced in Figure 2.

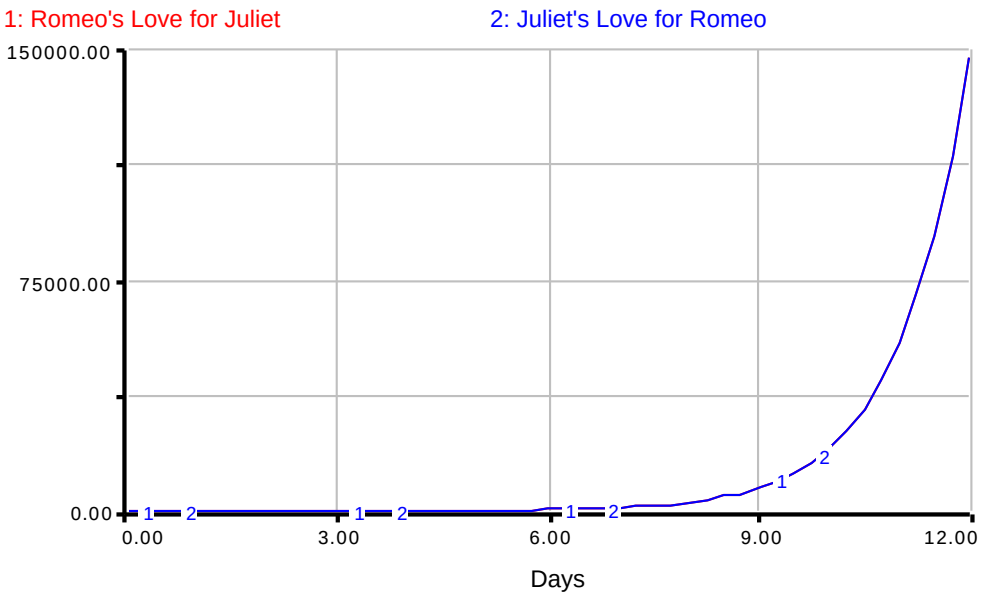

Figure 2: Romeo and Juliet initially like each other

Romeo's love and Juliet's love grow together exponentially. By the end of 12 days, they each feel 150,000 love units! In other words, they like each other 150,000 times more than they did initially.

Why does their love grow exponentially? The stock of Romeo's love for Juliet is positive and Juliet's reaction to Romeo's love is also positive, so the stock of Juliet's love increases. Similarly, because the stock of Juliet's love for Romeo is positive and Romeo's reaction to Juliet's love is also positive, the inflow increases the stock of Romeo's love. A positive feedback loop joins the two stocks. An increase in Juliet's love for Romeo increases the change in Romeo's love, which increases Romeo's love for Juliet. The increase in Romeo's love leads to an increase in the change in Juliet's love, further increasing Juliet's love for Romeo. With time, the stocks reinforce each other and increase exponentially.

What would happen if initially Romeo loved Juliet twice as much as she loved him? Will Juliet's love ever grow to equal Romeo's love? Mentally simulate the new model. Figure 3 displays the actual behavior.

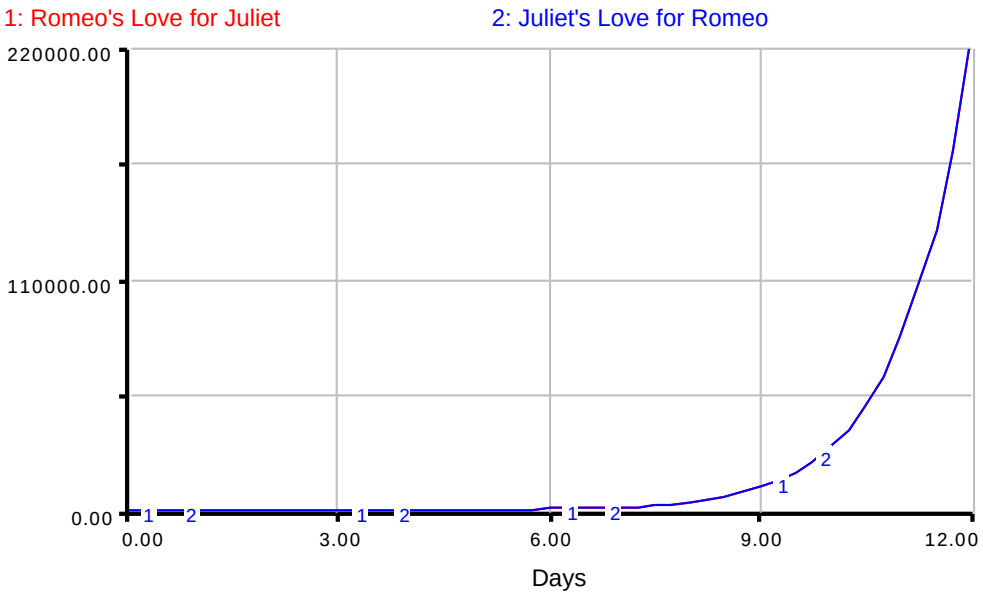

Figure 3: Initially Romeo likes Juliet twice as much as she likes him

Juliet's love quickly catches up with Romeo's love. The change in Juliet's love is proportional to Romeo's love. If Romeo initially loves Juliet twice as much, then Juliet's love increases twice as much as it would have increased if initially he only felt one unit of love. The additional increase precipitates the exponential growth of their love, so after 12 days they love each other even more than in the first scenario.

Consider the scenario in which Juliet is entirely indifferent to Romeo. Romeo, however, catches a glimpse of Juliet at a party and finds her really attractive. In other words, Juliet initially feels zero love units for Romeo, but Romeo feels one love unit for Juliet. Mentally simulate the model.

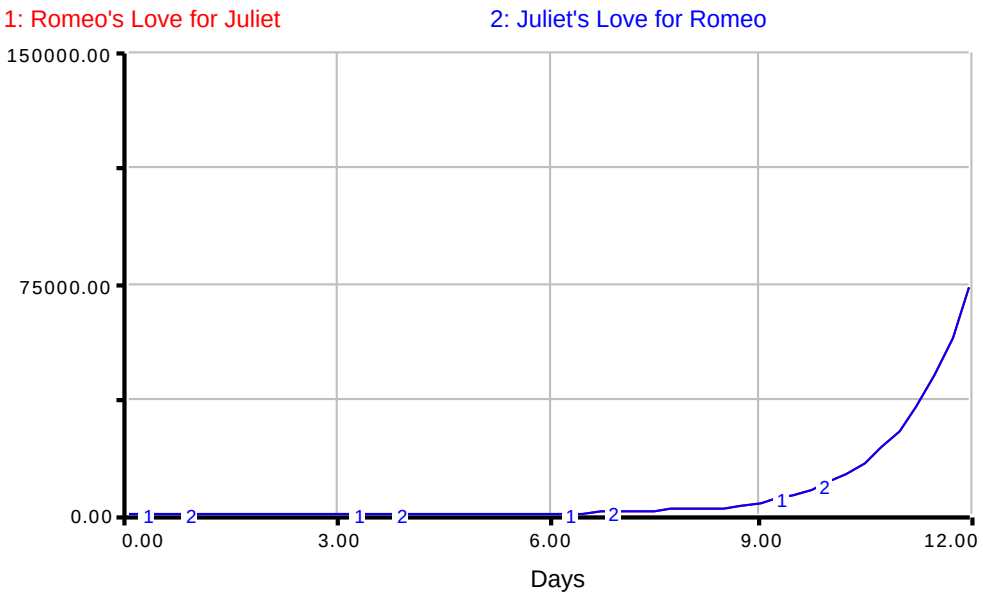

Figure 4: Initially Romeo likes an indifferent Juliet

Again, as seen in Figure 4, both stocks grow together exponentially. Because of Juliet's initial indifference, however, on the twelfth day, their love is "only" 75,000 times as strong as Romeo's initial reaction to Juliet. If Juliet had initially responded in the same way as Romeo, as in the first scenario, their love would have been twice as strong, as displayed in Figure 2. To summarize, if just one of the two lovers feels some positive emotion for the other, the love of both lovers will feed off of each other, and both stocks will grow exponentially.

Now imagine a scenario in which Romeo and Juliet are so influenced by the animosity between their families that when they first meet, each feels some instinctive dislike for the other. What will happen to them? Mentally simulate the model for the scenario in which initially both Romeo and Juliet feel one negative unit of love for each other.

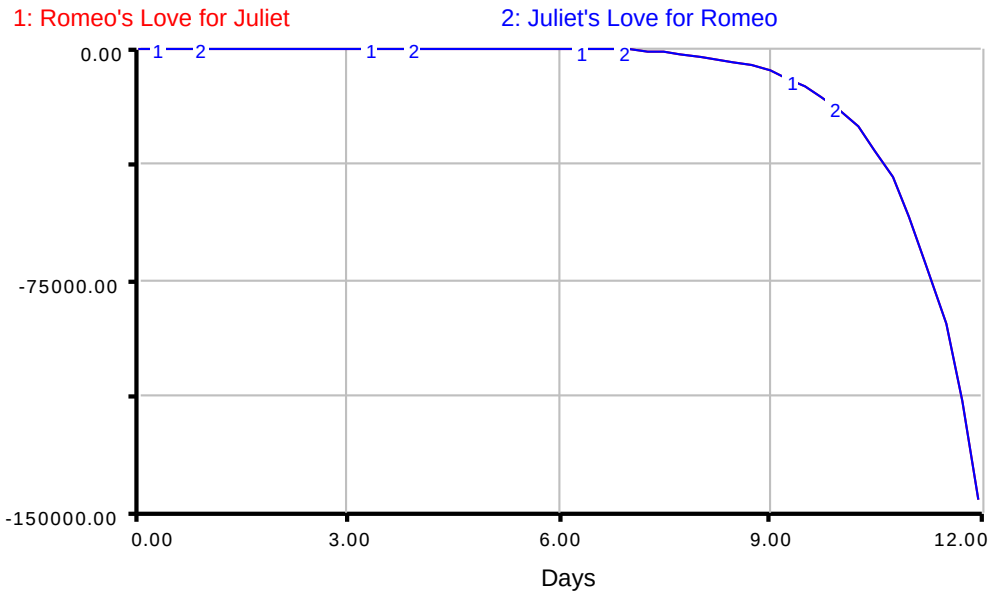

Figure 5: Initially Romeo and Juliet dislike each other

Figure 5 shows how Romeo and Juliet's dislike for each other experiences exponential decline and turns into adamant hatred. Juliet dislikes Romeo more because he dislikes her. He dislikes her more because she dislikes him. The positive feedback loop reinforces each character's opinion of the other. Instead of producing exponential growth towards an infinitely large positive value, the model now produces exponential decline towards an infinitely large negative value.

Now imagine a scenario in which, although Romeo was initially enchanted by Juliet, she felt an initial dislike for him. Let the initial value of Romeo's love be one positive love unit and the initial value of Juliet's love be one negative love unit. What will happen to their love? Mentally simulate the model.

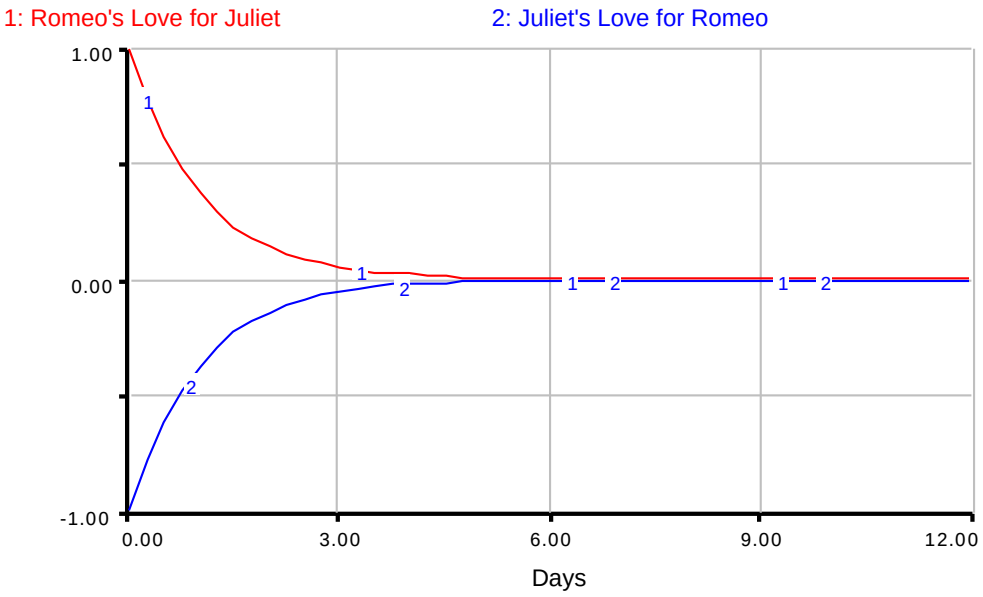

Figure 6: Initially Juliet and Romeo have opposite reactions to each other

On a first glance, the behavior in Figure 6 is very surprising. The feedback loop between the two stocks of love is positive, so one would expect to see the exponential behavior typically generated by positive feedback loops. In this particular situation, however, both stocks are formulated identically. Both stocks have the same initial values, and the same parameters regulate the flows. The stocks exactly balance each other and together decay asymptotically.

Trace through the model diagram. Juliet's dislike for Romeo decreases his love for her. As long as her love for him is negative, his love for her decreases. While his love for her is positive, however, her love for him increases, becoming less negative. As Juliet's love becomes less negative, Romeo's love still decreases, but by less. Eventually, they stabilize at total indifference towards each other.

Similarly, if initially Romeo and Juliet are initially entirely indifferent to each other, then the system will remain at equilibrium, as demonstrated in Figure 7.

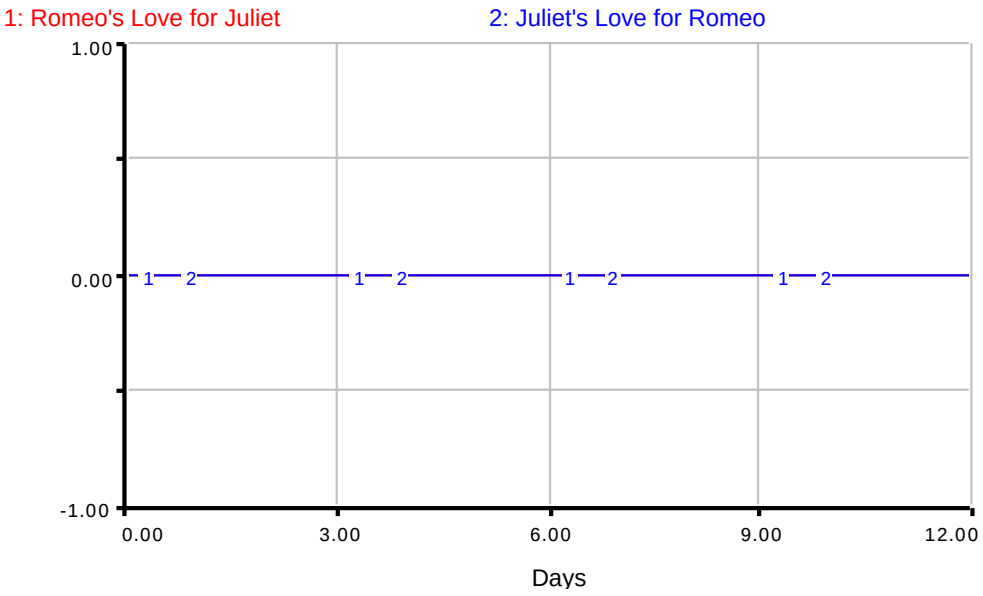

Figure 7: Initially Romeo and Juliet are indifferent to each other

The change in each person's love depends on the level of the other person's love. Because neither person loves the other, the rates of change are also zero, and the system never leaves the equilibrium state. Romeo and Juliet never click, so nothing happens.

The equilibrium that they reach, however, is very unstable. If, due to any external reason, either Romeo or Juliet stops feeling indifferent to the other, their love will once again grow or decline exponentially. Figure 8, for example, displays what happens if Juliet is indifferent to Romeo but Romeo feels a very slight inclination—a hundredth of a love unit—for Juliet.

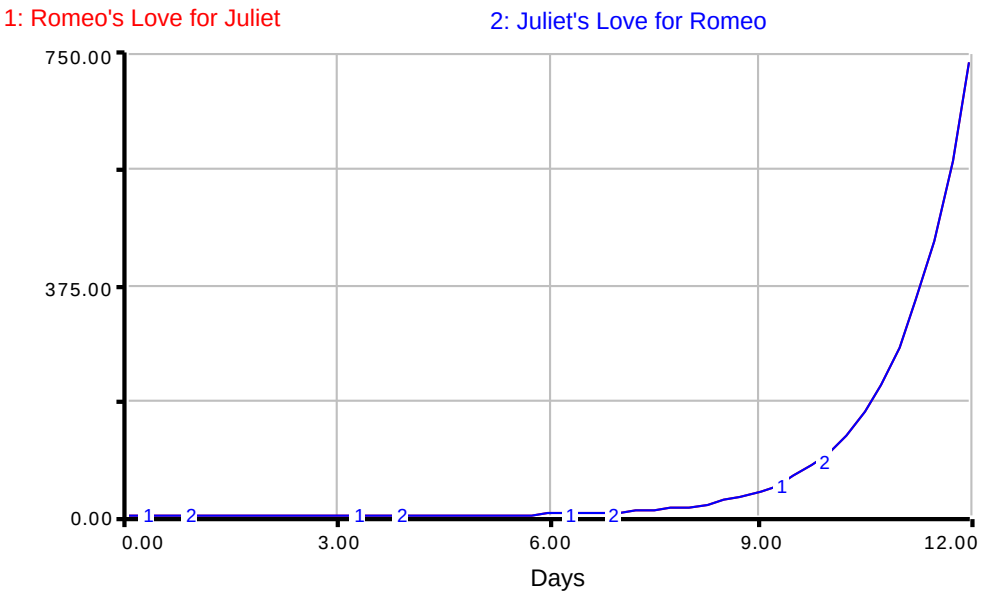

Figure 8: Initially Romeo feels 1/100 of a love unit for an indifferent Juliet

Any perturbation, that is, any change to either Romeo or Juliet's love, will cause both stocks to grow or decline exponentially.

4. The Red and the Black

The two protagonists in *The Red and the Black*,[4] Julien and Claire, have a love-hate relationship. Claire is attracted to Julien when he courts her friends and shows no interest in her. When Julien begins to love her, he becomes pathetic in her eyes, and she develops a disgust for him. Julien is likewise attracted to Claire only when she gives him a cold shoulder and treats him with contempt. Then she becomes the prize that his honor mandates that he catch.

What will happen when these two characters meet?

[4] from Stendhal, *The Red and the Black.*

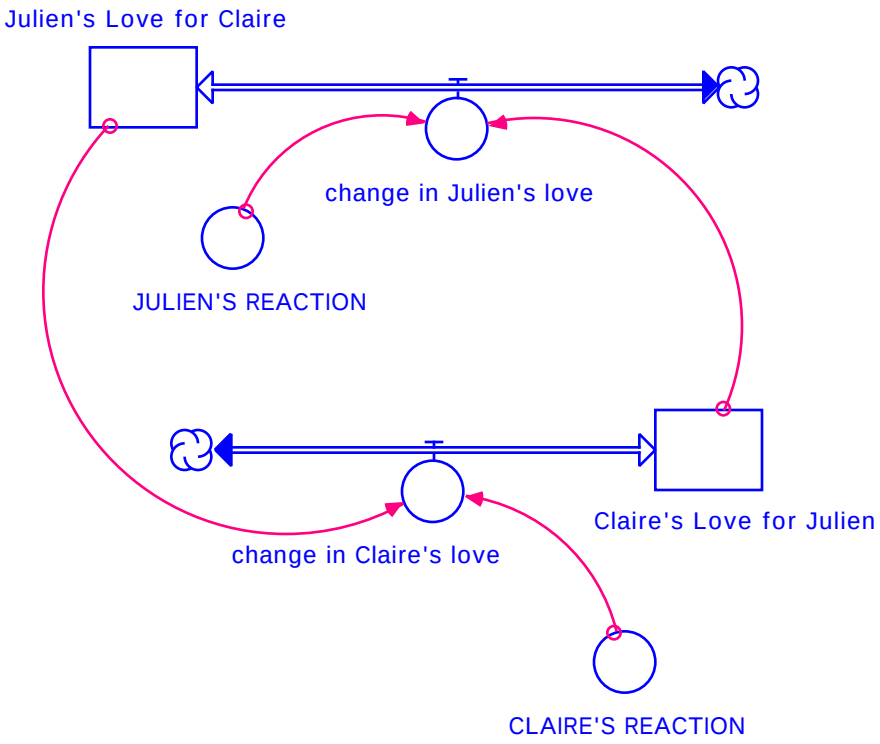

Figure 9: Model diagram for Julien and Claire

The model in Figure 9 represents the interactions of Julien and Claire. Both Julien's reaction to Claire's love and Claire's reaction to Julien's love are negative. Every unit of love that Julien displays to Claire decreases Claire's love for Julien by one unit, and vice versus.

In the novel, Julien initially likes Claire, but Claire dislikes him. Let Julien's love be one positive love unit and Claire's love be one negative love unit. Mentally simulate the model.

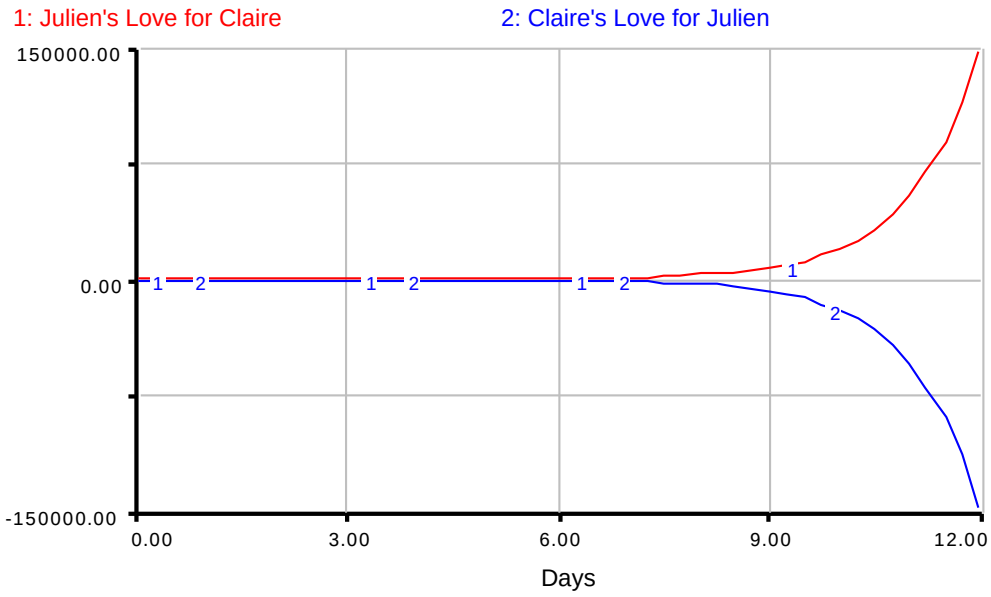

Figure 10: Initially Julien likes Claire but she dislikes him

As shown in Figure 10, the stocks diverge towards positive and negative infinity. Julien's love grows exponentially while Claire's love declines exponentially. Julien grows to adore Claire while she grows to loathe him. Her initial dislike arouses more love in Julien, which drives her to dislike him even more. The more she dislikes him, the more he loves her.

Is the feedback loop in *The Red and the Black* model positive or negative? One way to determine whether a feedback loop is positive or negative is to trace through the loop and see whether a change in one variable is reinforced or stabilized. An increase in Julien's love for Claire decreases Claire's love for Julien which further increases Julien's love for Claire. The change is compounded so the feedback loop is positive. Another way to determine the polarity of a feedback loop is to multiply the signs of all elements in the feedback loop. In the Romeo and Juliet model, both parameters are positive (and the flows are both positive) so the feedback loop is positive * positive = positive. In the Red and the Black model, both parameters are negative so the feedback loop is negative * negative = positive. As expected, the positive feedback loop generates exponential behavior.

Now consider the scenario in which initially Julien loves Claire a little more than Claire loves Julien. (Julien initially feels one positive love unit whereas Claire only feels 0.9 love units, that is, she likes him 10% less than he likes her). What will happen?

Mentally simulate the model. Running the simulation generates the behavior depicted in Figure 11.

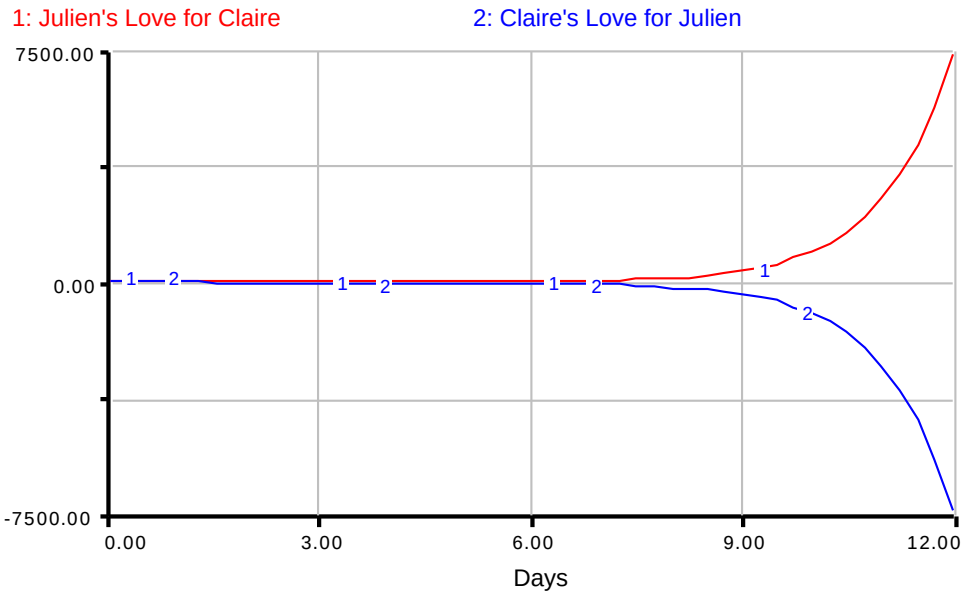

Figure 11: Initially Julien likes Claire a little more

Again Julien's love grows exponentially while Claire's love declines exponentially. At first the behavior in Figure 11 may seem surprising. Note the little blip in the curve during the first two days. Figure 12 zooms in to observe the dynamics at that blip.

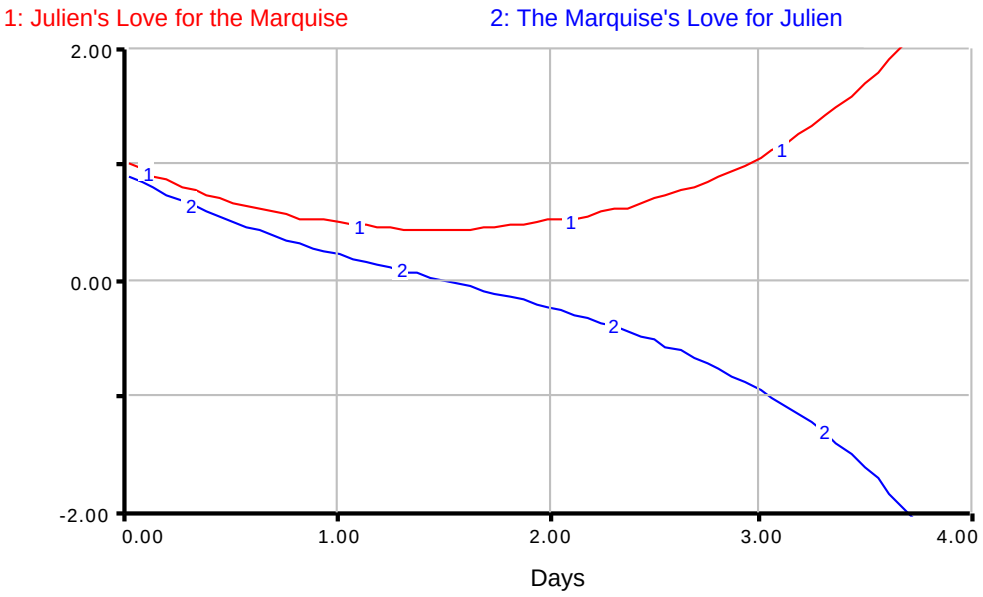

Figure 12: Initially Julien likes Claire a little more, detail

Initially, Claire's love is positive so Julien's love decreases. Julien's love is initially greater than Claire's love, so her love decreases by even more. As Claire loves Julien less, his love for her decreases at a slower rate. After about a day and a half, Claire's love becomes negative. Now that Claire dislikes him, Julien is more attracted to her. The more Julien likes her, the more Claire dislikes him. The reinforcing behavior causes Julien's love to grow exponentially and Claire's love to decline exponentially.

Note that the system does not stabilize at equilibrium because when Claire becomes indifferent to Julien, he still likes her. His love fuels Claire's hatred.

Now, consider the situation in which Julien and Claire at first both feel exactly the same way for each other. Each initially feels one unit of love for the other. Mentally simulate the model. Simulate the model in STELLA produces the behavior shown in Figure 13.

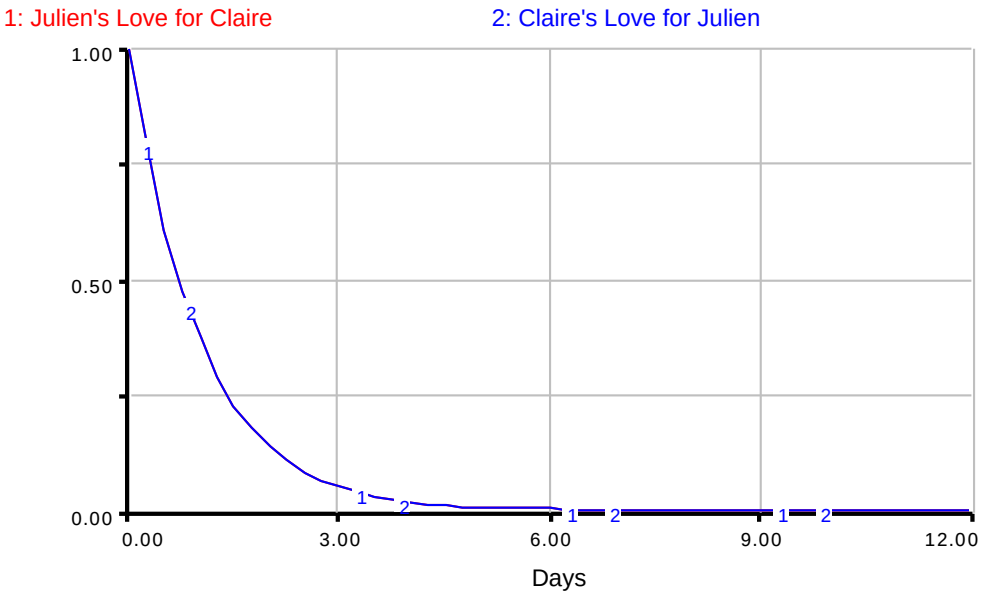

Figure 13: Initially Julien and Claire like each other

Why does the positive feedback loop generate asymptotic behavior? Again, this scenario is an exceptional case in which the two stocks are exactly balanced against each other. Trace through the model to understand the dynamics. Claire's love for Julien is positive so Julien's love for Claire decreases. Julien's love for Claire is also positive, so Claire's love for Julien also decreases. With time, however, Julien's love is decreasing, so Claire's love decreases by less and less. Likewise, Claire's love is decreasing so Julien's love decreases by less and less. As their love approaches zero, the change in their love becomes smaller and smaller until finally the system stabilizes at an equilibrium of zero, when both Julien and Claire are indifferent to the other. Note again that this equilibrium is unstable.

Now consider the scenario in which initially Claire and Julien dislike each other. Let the initial values of the stocks be one negative love unit. Mentally simulate the model. You should expect to see the behavior pictured in Figure 14.

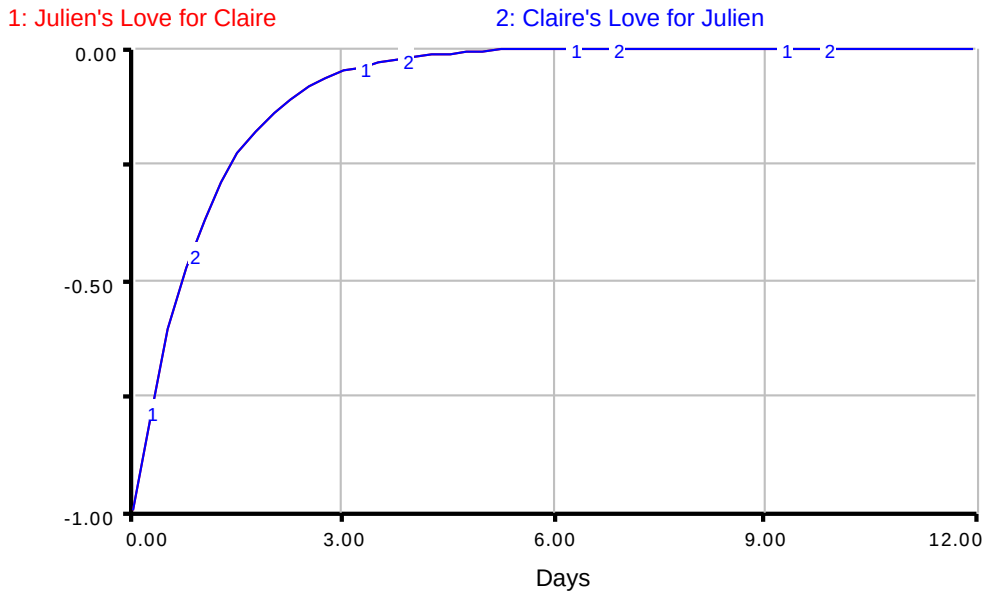

Figure 14: Initially Julien and Claire dislike each other

This scenario exhibits dynamics very similar to the dynamics of the previous scenario. Initially both Julien's love and Claire's love are negative, which makes each of their love for the other less negative. As each person dislikes the other less, he or she becomes less attractive to the other. As long as each character still dislikes the other, the other's love will continue to increase, but by less and less. Finally Julien and Claire reach an equilibrium of indifference.

Finally, consider the scenario in which Julien and Claire are both are indifferent to the other initially. You should be able to guess what will happen. Simulate the model generates the behavior reproduced in Figure 15.

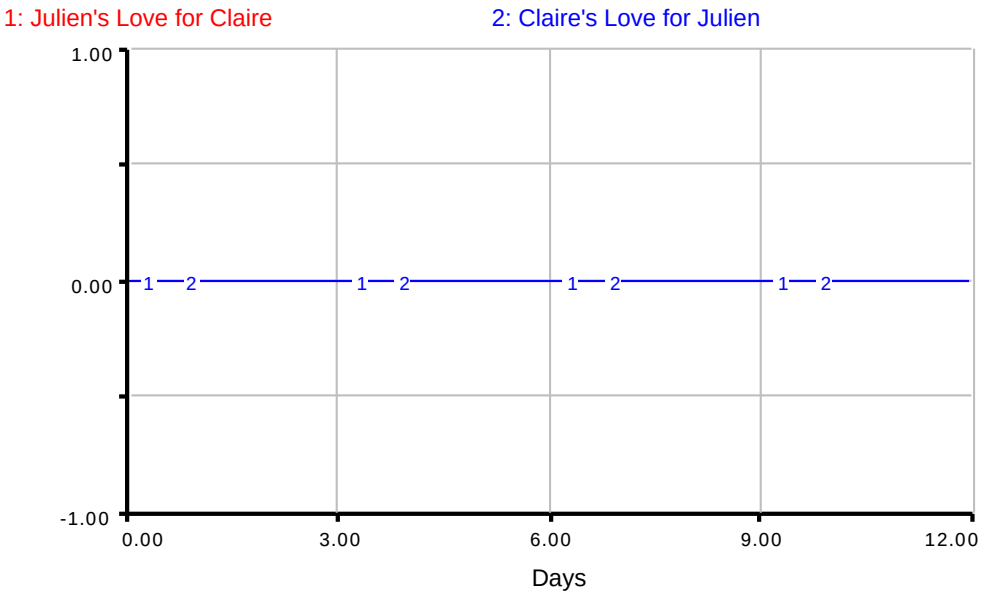

Figure 15: Initially Julien and Claire are indifferent to each other

As expected, the system starts off at equilibrium and remains at equilibrium. Claire and Julien leave each other alone.

5. Gone with the Wind

Scarlet[5] always wants what she cannot have. When Rhett is indifferent to her, she is attracted to him. When Rhett is attracted to her, she scorns him. Rhett, however, becomes frustrated when Scarlet treats him with contempt. He is attracted to her when she is attracted to him, but he loses interest when she snubs him.

What will happen when these two characters meet?

[5] from Margaret Mitchell, *Gone with the Wind.*

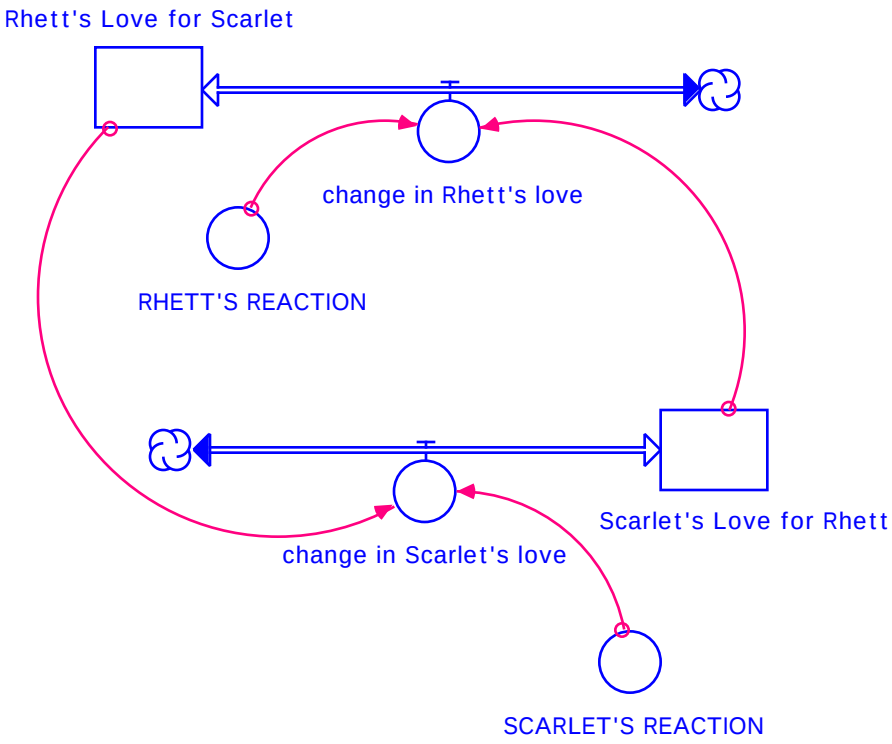

Figure 16: Model Diagram for Rhett and Scarlet

Figure 16 shows a model diagram describing Rhett and Scarlet's love. Rhett's reaction to Scarlet's love is positive, while Scarlet's reaction to Rhett's love is negative. Consider the scenario in which the two characters initially each feel one unit of love for the other. What do you think will happen to Rhett and Scarlet's love? Mentally simulate the model.

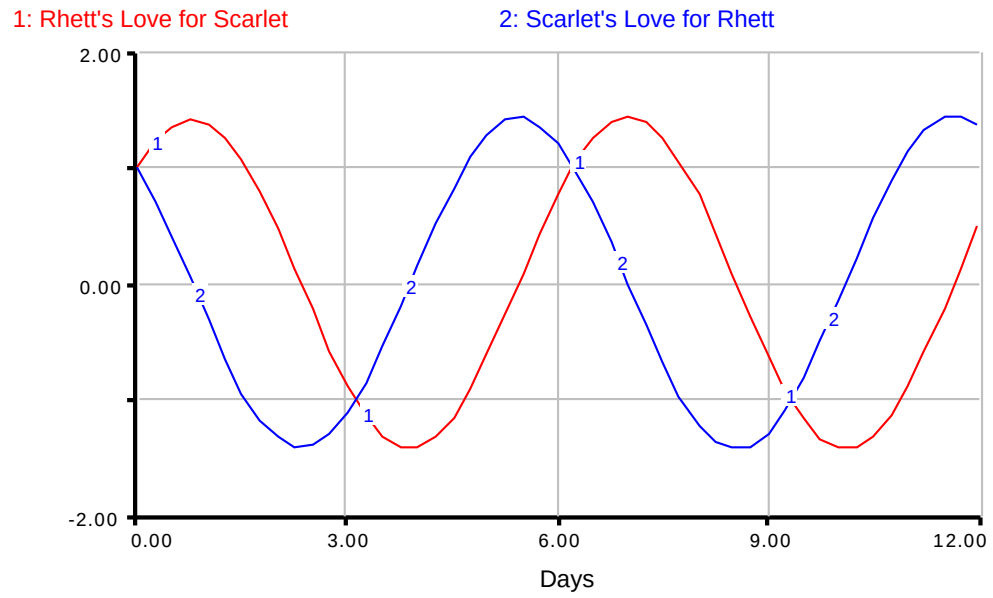

Figure 17: Rhett and Scarlet's love oscillates

Figure 17 shows that Rhett and Scarlet's love exhibits sustained oscillations. Initially Scarlet's love is positive so Rhett's love begins to increase. Because Rhett's love is positive, Scarlet's love decreases. When Scarlet's love turns negative, Rhett's love for her begins to decrease. When his love for her becomes negative, she experiences a change of heart, and her love for him increases. When her love turns positive, Rhett changes his mind, and his love begins to increase again. When his love becomes positive, Scarlet reassesses her opinion of him, and her loves once again decreases. The two lovers continue in this oscillatory pattern.

What is the polarity of the feedback loop in the Gone with the Wind model? First, trace through the feedback loop. If Rhett's love increases, Scarlet's love either decreases or increases by less. If Scarlet's love decreases or increases by less, then Rhett's love will either decrease or increase by less, acting to counterbalance the initial increase in Rhett's love. The feedback loop is therefore stabilizing, that is, negative. Check by multiplying the signs of the elements in the feedback loop. One parameter is positive and the other is negative. The feedback loop is therefore positive * negative = negative.

Now consider the scenario in which Rhett loves Scarlet but Scarlet is indifferent to Rhett. How will the behavior in this scenario be different from the behavior in the previous scenario? Mentally simulate the model. You should expect to observe the behavior depicted in Figure 18.

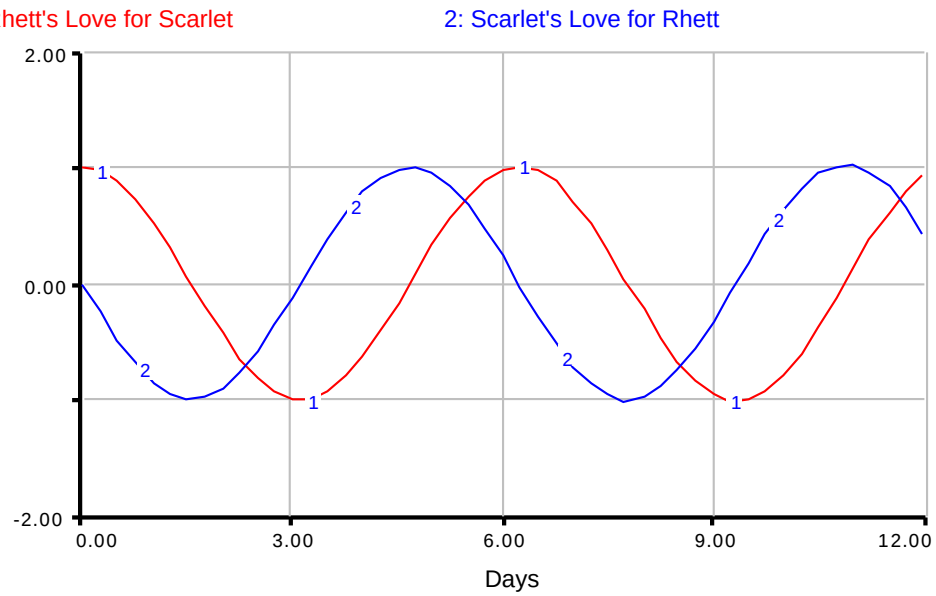

Figure 18: Rhett likes Scarlet but she is indifferent

Scarlet is immediately turned off by Rhett's feelings for her. Scarlet's feelings for Rhett plunge from indifference to dislike. So right away Rhett's love for Scarlet begins to decrease. His love vanishes faster than in the first scenario, in which his love had grown the entire time Scarlet's love had been positive (albeit decreasing). When his love becomes negative, Scarlet begins to dislike him less. Because Rhett's love becomes negative earlier than it did in the first scenario, Scarlet's love does not have the time to decrease as far as it did in the first scenario before she begins to regain interest in him. The amplitude of the oscillations is smaller. The cycle then continues, as Scarlet only loves Rhett when he dislikes her, and he only likes her when she likes him back.

Now imagine a scenario in which initially Rhett likes Scarlet but Scarlet actively dislikes Rhett. What will happen to the two lovers?

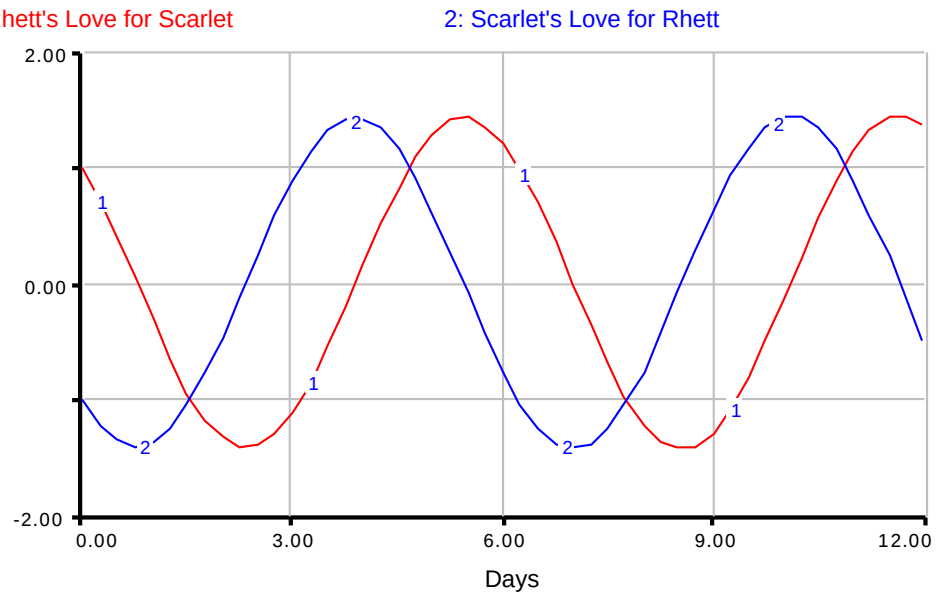

Figure 19: Rhett likes Scarlet but she dislikes him

The oscillations in Figure 19 are very similar to the oscillations in the first scenario of this section. As long as Rhett's love for Scarlet is positive, her love decreases. When his love becomes negative, her love begins to increase. When her love becomes positive, his love starts becoming less negative. When his love becomes once again positive, her love for him begins to decrease. It seems as though there is no way for the two lovers to both be happy.

Finally, consider the scenario in which Rhett and Scarlet are initially indifferent to each other. Will the system still oscillate? Simulating the model generates the behavior pictured in Figure 20.

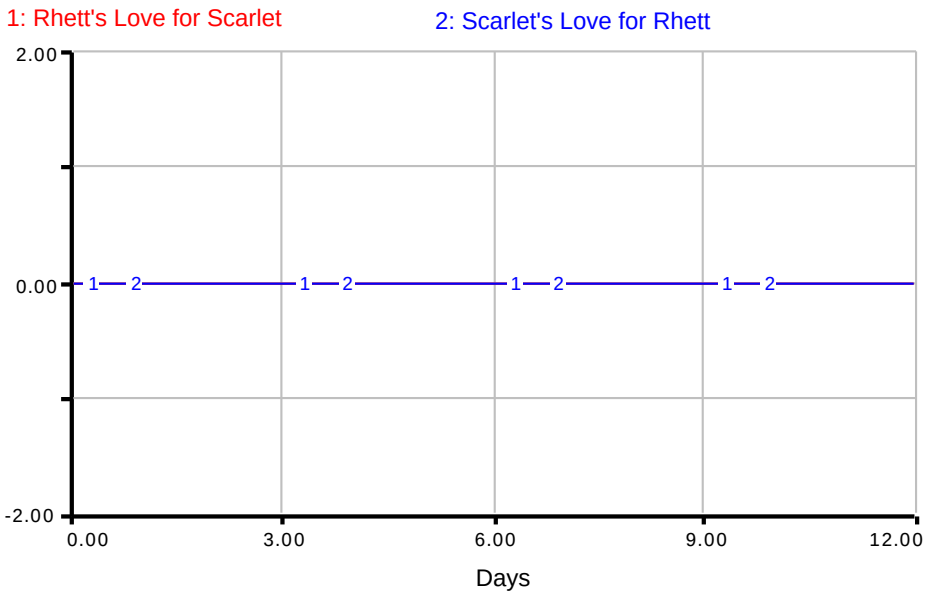

Figure 20: Rhett and Scarlet are indifferent to each other

The net flow into the stocks is zero because the stocks are initially zero. If the net flow is zero then neither stock changes and the net flow into each stock is always zero. If they are both indifferent, nothing will ever attract Rhett and Scarlet to each other or repel them from each other.

6. Conclusion

The structure of the *Romeo and Juliet*, *The Red and the Black*, and the *Gone with the Wind* models is very similar to the structure presented in the two generic structures papers in Road Maps that describe sustained oscillations. The generic structure for sustained oscillations is reproduced in Figure 21.

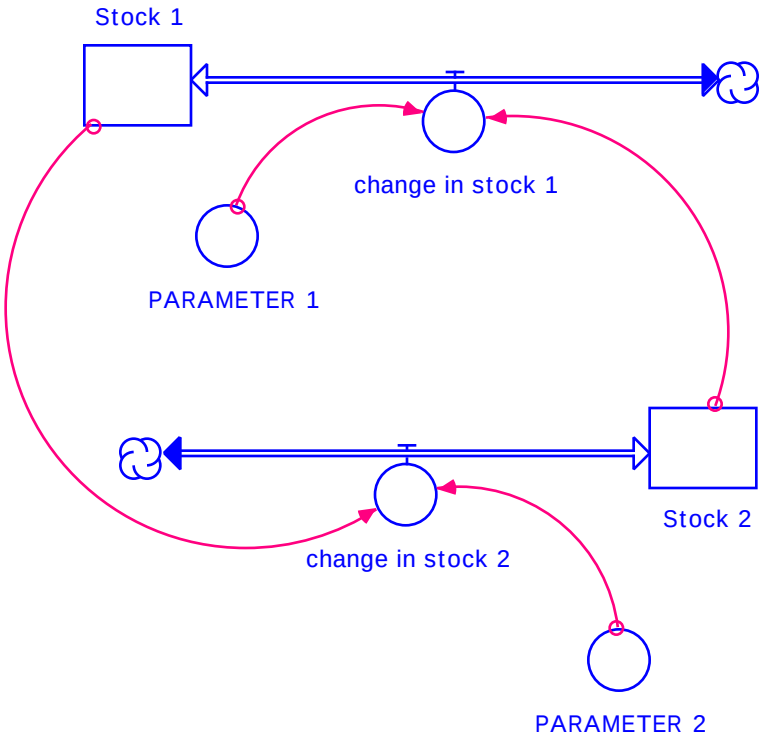

Figure 21: Generic structure for sustained oscillations

Although the structure in Figure 21 looks similar to the structure of the Romeo and Juliet and The Red and the Black models, no set of initial conditions can make the latter models oscillate. Oscillations are a feature of *negative* feedback loops in higher-order (two stocks or more) systems. The Romeo and Juliet and The Red and the Black models are driven by a positive feedback loop, which usually generates exponential behavior but can, in some exceptional cases, generate asymptotic behavior.

7. Appendix

7.1 Equations for the Romeo and Juliet model

☐ **Romeo's Love for Juliet(t)** = Romeo's Love for Juliet(t - dt) + (change in Romeo's love) * dt
INIT Romeo's Love for Juliet = *depends on scenario*
DOCUMENT: the love Romeo has for Juliet
UNIT: love units

☐ **Juliet's Love for Romeo(t)** = Juliet's Love for Romeo(t - dt) + (change in Juliet's love) * dt
INIT Juliet's Love for Romeo = *depends on scenario*

DOCUMENT: the love Juliet has for Romeo
UNIT: love units

change in Romeo's love = Juliet's Love for Romeo * ROMEO'S REACTION
DOCUMENT: Romeo's love only changes in proportion to Juliet's love
UNIT: love units / day

change in Juliet's love = Romeo's Love for Juliet * JULIET'S REACTION
DOCUMENT: Juliet's love only changes in proportion to Romeo's love
UNITS: love units / day

ROMEO'S REACTION = 1
DOCUMENT: the effect of Juliet's love on Romeo's love
Units: 1/day

JULIET'S REACTION = 1
DOCUMENT: the effect of Romeo's love on Juliet's love
UNITS: 1/day

7.2 Equations for the Red and the Black model

Julien's Love for Claire(t) = Julien's Love for Claire(t - dt) + (change in Julien's love) * dt
INIT Julien's Love for Claire = *depends on scenario*
DOCUMENT: the love Julien feels for Claire.
UNITS: love units

Claire's Love for Julien(t) = Claire's Love for Julien(t - dt) + (change in Claire's love) * dt
INIT Claire's Love for Julien = *depends on scenario*
DOCUMENT: the love Claire feels for Julien.
UNITS: love units

change in Julien's love = Claire's Love for Julien * JULIEN'S REACTION
DOCUMENT: Julien's love only changes in proportion to Claire's love
UNIT: love units / day

change in Claire's love = Julien's Love for Claire * CLAIRE'S REACTION
DOCUMENT: Claire's love only changes in proportion to Julien's love
UNIT: love units / day

JULIEN'S REACTION = -1
DOCUMENT: the effect of Claire's love on Julien's love
UNITS: 1/day

CLAIRE'S REACTION = -1
DOCUMENT: the effect of Julien's love on Claire's love
UNITS: 1/day

7.3 Equations for the Gone with the Wind model

□ **Rhett's Love for Scarlet(t)** = Rhett's Love for Scarlet(t - dt) + (change in Rhett's love) * dt
INIT Rhett's Love for Scarlet = *depends on scenario*
DOCUMENT: the love Rhett feels for Scarlet
UNITS: love units

□ **Scarlet's Love for Rhett(t)** = Scarlet's Love for Rhett(t - dt) + (change in Scarlet's love) * dt
INIT Scarlet's Love for Rhett = *depends on scenario*
DOCUMENT: the love Scarlet feels for Rhett
UNITS: love units

change in Rhett's love = Scarlet's Love for Rhett * RHETT'S REACTION
DOCUMENT: Rhett's love only changes in proportion to Scarlet's love
UNIT: love units / day

change in Scarlet's love = Rhett's Love for Scarlet * SCARLET'S REACTION
DOCUMENT: Scarlet's love only changes in proportion to Rhett's love
UNIT: love units / day

○ **RHETT'S REACTION** = 1
DOCUMENT: the effect of Scarlet's love on Rhett's love
UNITS: 1/day

○ **SCARLET'S REACTION** = -1
DOCUMENT: the effect of Rhett's love on Scarlet's love
UNITS: 1/day