UNDERSTANDING OSCILLATIONS IN SIMPLE SYSTEMS

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I. INTRODUCTION

Acquiring a firm intuitive understanding of the possible types of behavior produced by simple first, second, and third-order system marks an important step in learning system dynamics. Such simple systems frequently embody generic structures that recur in a wide variety of complex systems. However, an intuitive grasp of simple oscillating systems often eludes both the beginning student and practitioner alike. Even individuals familiar with the mathematics of dynamic feedback systems often cannot provide a simple nontechnical explanation of why a continuous first-order system cannot possibly oscillate, or why a second-order system can. For example, overshoot or oscillation in a system is often explained to result from "system delays" or "inertia." Vague explanations such as these impart little understanding of how decisions being made in a system generate observed problems and behavior.

This paper presents some arguments we have used in past introductory courses in system dynamics to successfully develop insight into simple oscillating systems. The paper analyzes a one-level model for the population growth of rabbits in a closed field to illustrate why a first-order negative-feedback system exhibits a smooth transition to equilibrium instead of overshoot or oscillating behavior. The paper also analyzes a simple inventory-workforce model to provide an intuitive explanation of the causes of convergent, divergent, and undamped oscillations.

II. WHY A FIRST-ORDER SYSTEM CANNOT OSCILLATE

The impossibility of oscillations in a first order system can be seen in the simple "rabbits in the field" model often used to introduce students to system dynamics. The model has a single level representing the population of rabbits R in a limited land area. The rabbits tend to grow exponentially until they begin to approach the carrying capacity of their environment. One possible model for this process contains a single positive-feedback loop controlling rabbit birth rate RBR and a single negative feedback loop controlling rabbit death rate RDR. Multiplying the normal number of litters born to each female rabbit per year (denoted litters per female-year LPFY), the normal number of surviving infants per litter SIPL, and the fraction of productive females in the population PFP yields a constant fractional birth rate FBR per year. Equations II.3 and II.4 describe the rabbit birth rate. The rabbit death rate RDR is computed by dividing the rabbit population R by the average lifetime of rabbits ALR. The average lifetime of rabbits diminishes as the rabbit population increases. This decline in average lifetime represents the effects of crowding and competition for food on mortality. As rabbit population increase, the average lifetime of rabbits falls until it eventually equals 1/FBR, causing the rabbit birth and death rates to be equal.[1] Equations II.5 through II.7 determine rabbit death rate RDR in the model.

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1. Note that when the average lifetime equals 1/FBR, rabbit death rate = Rabbits/average rabbit lifetime = R/(1/FBR) = (R)(FBR) = rabbit birth rate.

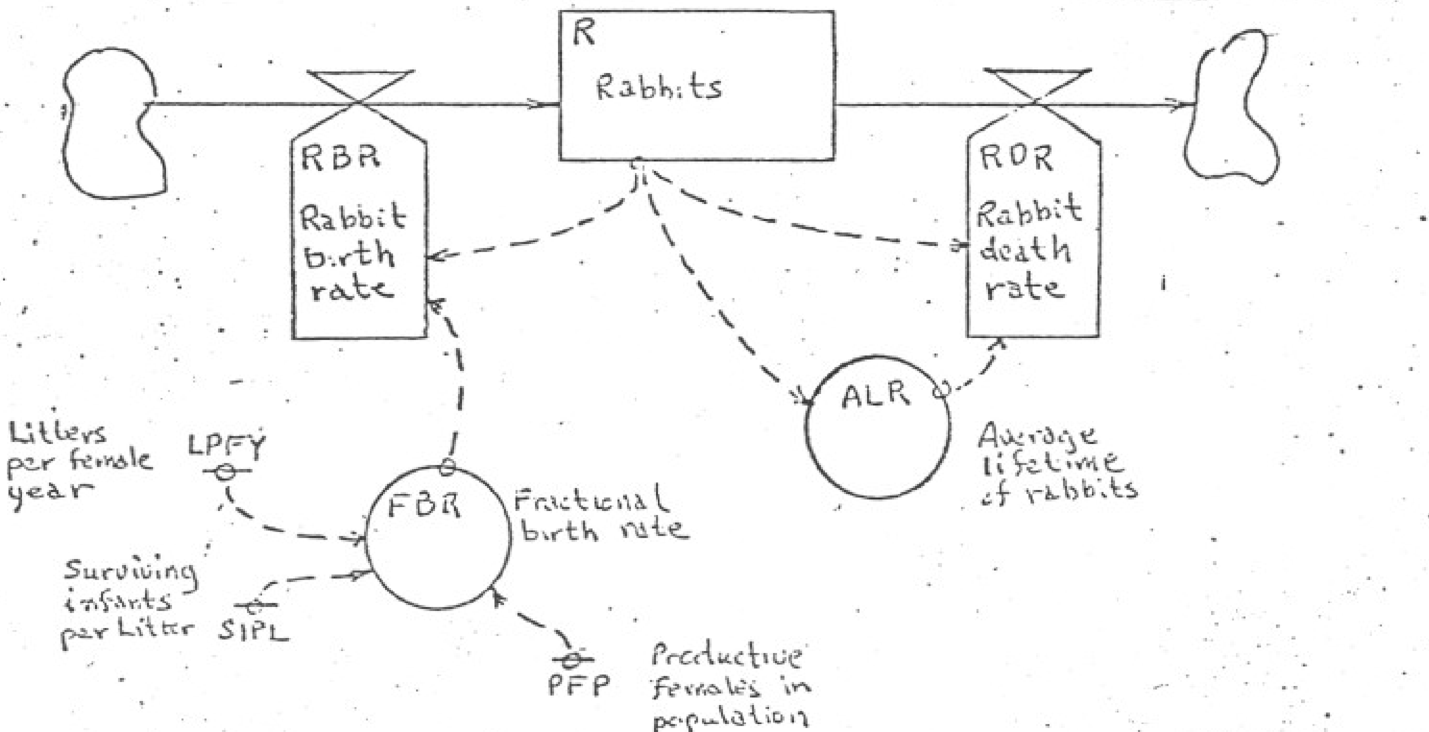

Figure 1. Flow Diagram of First-Order Rabbit Model

II.1	L R.K = R.J + DT * (RBR.JK - RDR.JK)	Rabbit population (rabbits)
II.2	N R = RI	Initial rabbit population (rabbits)
II.3	R RBR.KL = (R.K)(FBR.K)	Rabbit birth rate (rabbits/year)
II.4	A FBR.K = (LPFY)(SIPL)(PFP)	Fractional birth rate (1/year)
II.5	R RDR.KL = (R.K)(ALR.K)	Rabbit death rate (rabbits/year)
II.6	A ALR.K = TABHL(ALRT, R.K, 0, RMAX, INC)	Average lifetime of rabbits (years)
II.7	T ALRT = downward-sloping table	ALR as a function of R

(minimum value of ALR $\leq$ 1/FBR)

II.8	C LPFY =	Litters per female year (litters/rabbit/year)
II.9	C SIPL =	Surviving infants per litter (Rabbits/litter)
II.10	C PFP =	Productive females in population (dimensionless)

When asked what types of behavior can be expected from the one level rabbit model, students often respond that the level of rabbits will rise until it overshoots the carrying capacity of the finite environment and, then, oscillate about that carrying capacity before equilibrium is achieved. However, it can be shown readily that such behavior is impossible in the first-order system. Assume that the rabbit population did overshoot, as shown in Figure 2.

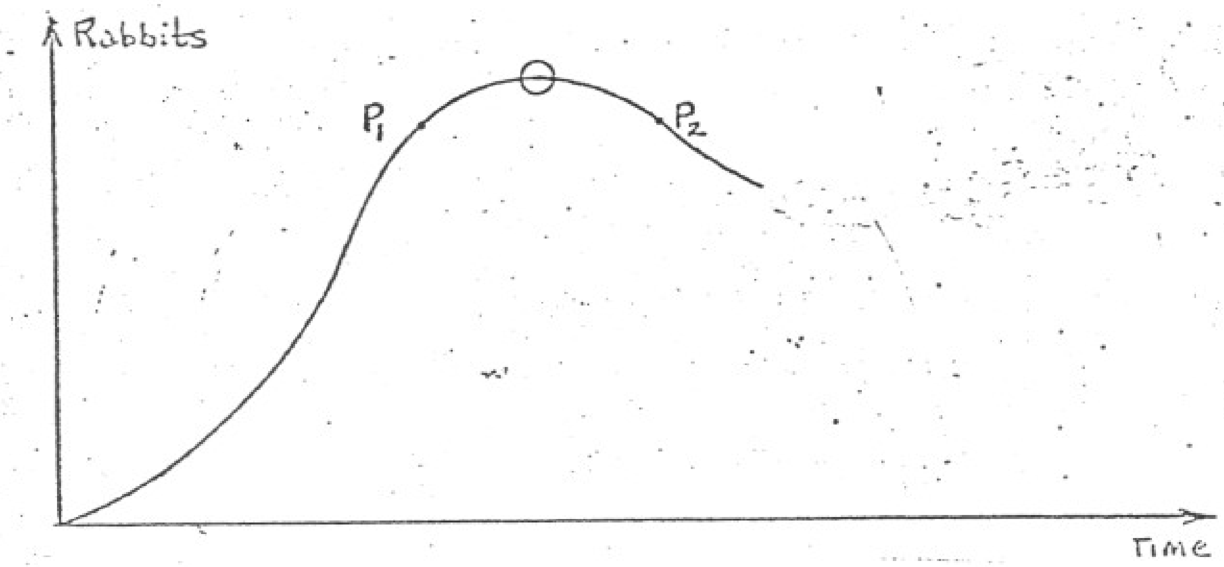

Figure 2. Suggested Overshoot and Oscillation for One-Level Rabbit Model

On the initial upswing, rabbit birth rate exceeds rabbit death rate and the population increases. As the limited space is filled, reduced food supply and other effects of overcrowding continuously reduce the average rabbit lifetime, causing rabbit death rate to gradually catch up to birth rate. If the overshoot in population is to occur, there must be some point at which births and deaths are balanced and population momentarily stops growing. This point is indicated on Figure 2 by a circle. However, once this point of "temporary equilibrium" is reached the first-order system cannot move from it. Birth rate and death rate can only vary if population varies. Moreover, population can only vary if there is an imbalance between the birth rate and the death rate. Once birth and death rate come into balance in the first-order system, the balance cannot be tipped and the system is locked into equilibrium.

Continuing this line of analysis shows that a second level or state variable is necessary for any form of oscillation to occur. Look, once again, at the circled peak in the suggested mode of behavior for the first-order system. Rabbit population will only start to fall if death rate rises above birth rate. In order for the balance between birth rate and death rate to be upset, there must be another variable in the system, influencing either births or deaths, <u>which continues to change even though rabbit population is temporarily unchanging</u>. This additional variable might be a predator population or a changing food supply. However, if this additional variable is to tip the balance between rabbit births and deaths, it cannot be an algebraic function of rabbit population (i.e. an auxiliary function of R), or it would remain constant when population is constant. If rabbit population is to overshoot, there must be an additional level or state variable in the system which continues to change when births equal deaths and causes deaths to rise above births.[2]

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2. There is an alternative way to see that two or more state variables are always required to describe an oscillating system. Consider the points P_1 and P_2 in Figure 2. At both points P_1 and P_2 the value of the rabbit population is the same. Yet, at P_1 births exceed deaths and at P_2 deaths exceed births. Some additional aspect of the state of the rabbit system must have changed as the system moved from P_1 and P_2. This additional information necessary to describe the internal state of the oscillating rabbit system implies that there must be at least one more state variable, in addition to rabbit population, present in the system.

III. UNDAMPED OSCILLATIONS IN A SECOND-ORDER SYSTEM

Section II showed that a system must have at least two levels to exhibit oscillatory behavior. However, the presence of two or more levels in a system does not guarantee oscillation. In order to understand why oscillations occur in a particular system, one must study the structure of that system. He must consider carefully what policies within the system lead to overshoot and oscillation instead of smooth adjustment to equilibrium. As an illustration of the type of understanding which should be strived for, section III explains why a simple two-level inventory-workforce system shows sustained oscillations.

The structure and equations for the inventory-workforce model are shown in Figure 3. The level of inventory INV is increased by production rate and reduced through sales rate (shipments). Workforce WF is changed by the hire-fire rate HFR, a two-way flow representing net additions to or subtractions from the firm's labor force. It is assumed in the model that production rate is directly proportional to workforce; there is a constant productivity per man PPM expressed in units/man-month (see equations III.3 and III.4). Moreover, it is assumed that the fim adjusts workforce according to the difference between inventory INV and some constant desired inventory DINV; the firm attempts to recruit new workers when inventory is below desired inventory, and contract employment when inventory is perceived to be excessive. Equation III.8 for the hire-fire rate HFR can be explained as follows. Suppose that the firm attempts to adjust inventory to desired inventory over some inventory adjustment time IAT. Then, desired inventory production, expressed in units/month, is equal to the inventory discrepancy, (DINV-INV), divided by the inventory adjustment time IAT. Now the desired labor increment (decrement) needed for inventory production is just the desired inventory production rate, [(DINV-INV)/IAT], divided by the productivity per man PPM; that is, [(DINV-INV)/IAT]/PPM. This term has the dimensions of

$$(\text{units/month})/(\text{units/man-month}) = \text{men}.$$

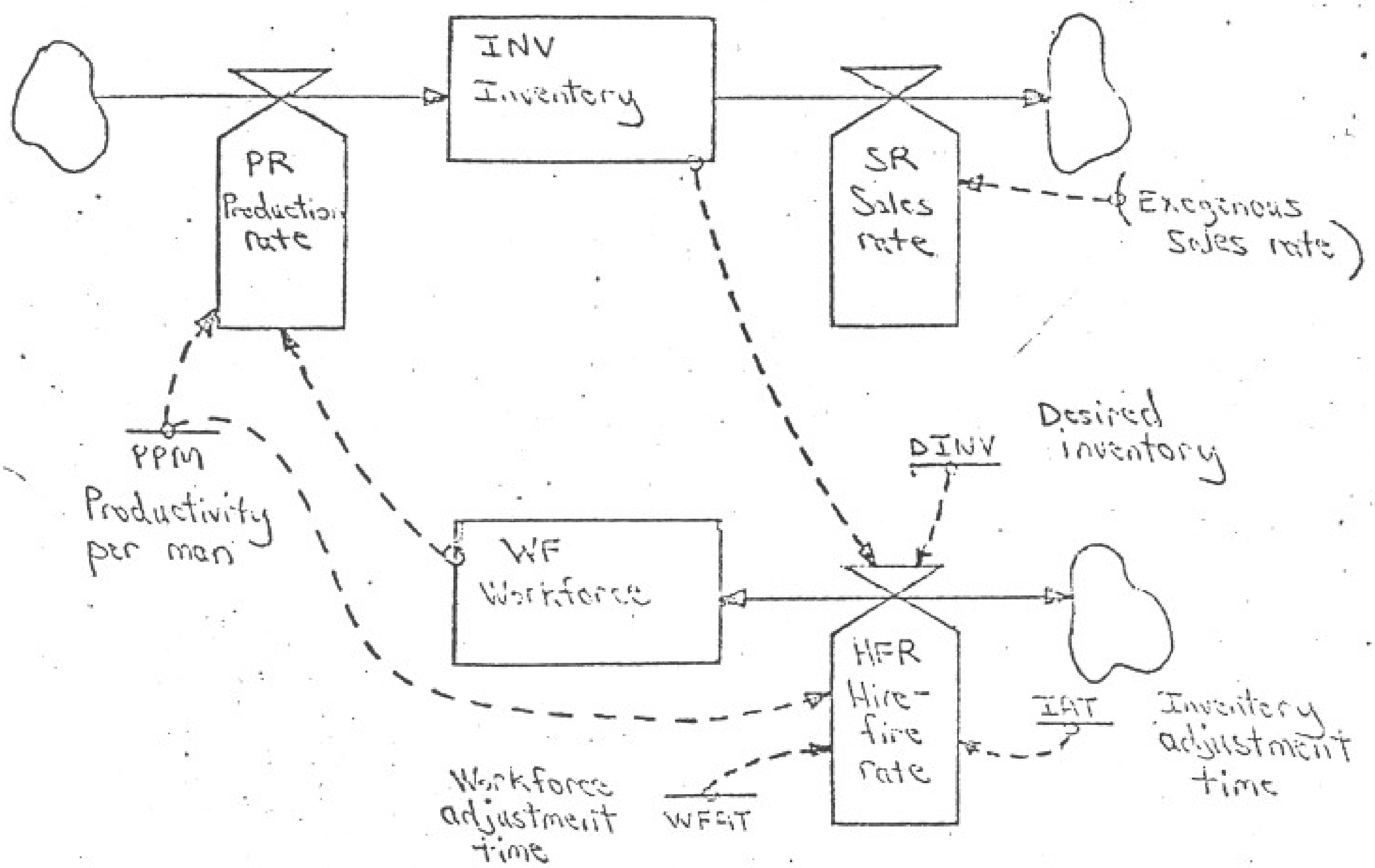
Dividing the desired labor increment by the workforce adjustment time WFAT yields the hire-fire rate:

$$HFR.KL = (DINV.K-INV.K)/(IAT)(PPM)(WFAT)$$

For simplicity, the hire-fire rate can be rewritten

$$HFR.KL = CHF(DINV.K-INV.K)$$

Where the coefficient for hires and fires CHF equals the reciprocal of the product of inventory adjustment time IAT, productivity per man PPM, and work-force adjustment time WFAT; CHF represents the number of men hired (or fired) each month in response to a one unit discrepancy in inventory. The above equation shows that HFR is proportional to the inventory discrepancy (DINV-INV).



III.1	L	INV.K = INV.J + (DT)(PR.JK - SR.JK)	Inventory (units)
III.2	N	INV = DINV	
III.3	R	PR.KL = WF.K * PPM	Production rate (units/month)
III.4	C	PPM =	Productivity per man (units/man-month)
III.5	R	SR.KL =	Exogenous sales rate (units/month)
III.6	L	WF.K = WF.J + (DT)(HFR.JK)	Workforce (men)
III.7	N	WF = SR/PPM	
III.8	R	HFR.KL = (DINV - INV.K)/(IAT * PPM * WFAT)	Hire-fire rate (men/month)
III.9	C	DINV =	Desired inventory (units)
III.10	C	IAT =	Inventory adjustment time (months)
III.11	C	WFAT =	Workforce adjustment time (months)

Figure 3. Second-Order Inventory-Workforce Model

The inventory-workforce system described in Figure 3 is initially in equilibrium since inventory equals desired inventory (equation III.2) and production rate equals sales rate (see equations III.3 and III.7). What will be the response of the inventory-workforce system if the sales rate is now increased? In particular, how will inventory behave over time? Students generally suggest the three modes shown in Figure 4 for the behavior of inventory--convergent, divergent, and undamped oscillations--with convergent oscillations being by far the most frequent choice. In fact, the system can exhibit only undamped oscillations no matter what parameter values are chosen for PPM, IAT, DINV, and WFAT. This result is often surprising to students since they perceive that the model firm is striving to adjust workforce to equalize actual and desired inventory, and they expect this policy to eventually equilibrate the system. The discussion below provides a non-technical explanation for why equilibrium can never be reached in this model once the system is perturbed.

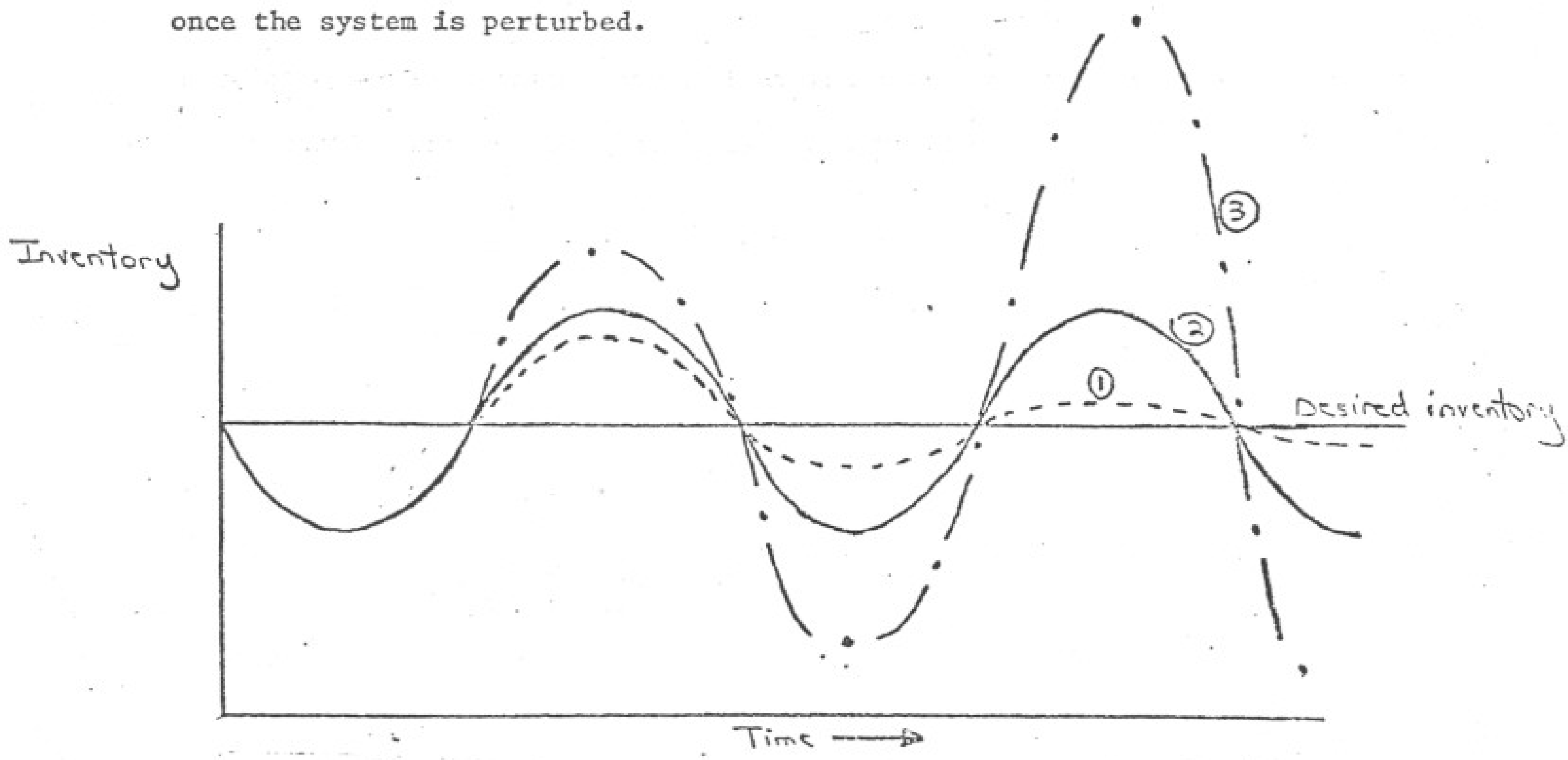


Figure 4. Suggested Response of Inventory to an Increase in Sales Rate

Suppose, as shown in Figure 5, that the inventory-workforce model is initially in equilibrium, with production rate equal to sales rate and inventory equal to desired inventory. Also assume that the sales rate is stepped upward at time t_0. What happens to inventory? As seen in Figure 5b, inventory starts to decline below desired inventory because the sales rate exceeds production rate. As this occurs, hiring is increased according to Equation III.8, in an effort to augment the firm's workforce. Thus, workforce and production rate (which is proportional to workforce) gradually rise. Inventory, however, continues to decline (although at a diminishing rate) as long as production rate is below sales rate. Inventory falls, rapidly at first, and more gradually as production rate rises near sales. In contrast, the hire-fire rate, which is proportional to the difference between actual and desired inventory, is largest at the point of minimum inventory; that is, at the point where production just equal sales. Thus, up to time t_1, when production rate equals sales rate, workforce and production rate expand at an increasing rate. The behavior of production rate up to time t_1 is shown in Figure 5a.

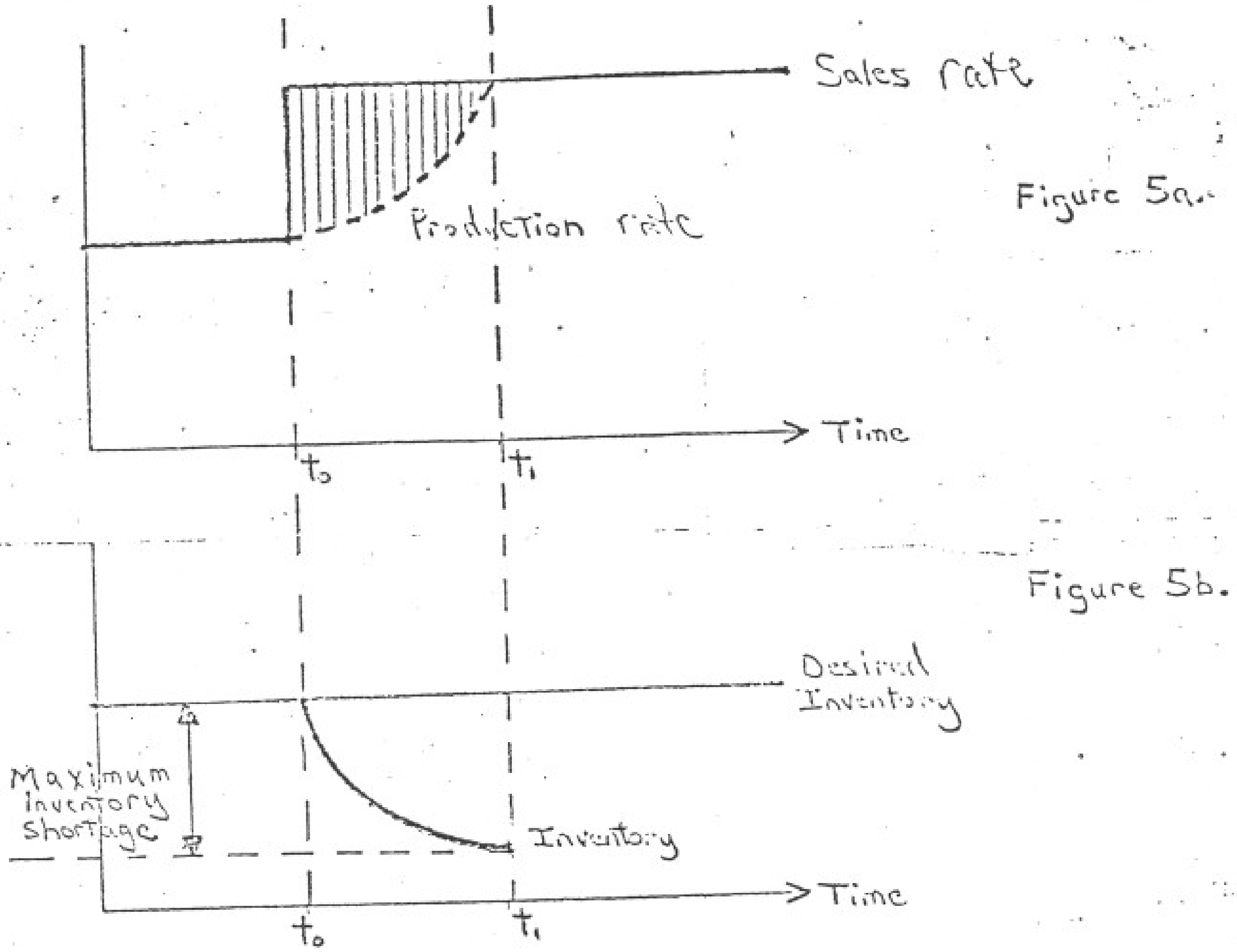


Figure 5. Increase of Production and Decline in Inventory in Response to an Increase in Sales

At time t_1, production rate now equals sales rate. But inventory is below desired inventory. The total loss of inventory between t_0 and t_1, shown in Figure 5b as the maximum inventory shortage, is given by the shaded area in Figure 5a. Because inventory is below desired inventory at time t_1, hiring continues to expand, thereby raising workforce and production rate. After time t_1, therefore, as shown in Figure 6a, production rate is above sales rate and the level of inventory starts to rise. Inventory rises gradually at first as production is only slightly above sales, and increases most rapidly when production rate is at its maximum value. In contrast to the behavior of inventory, hiring and production increase most rapidly initially, and later rise more gradually, since the difference between actual and desired inventory is narrowing over time. Figure 6 shows the behavior of inventory and production rate up to time t_2 when inventory once again equals desired inventory.

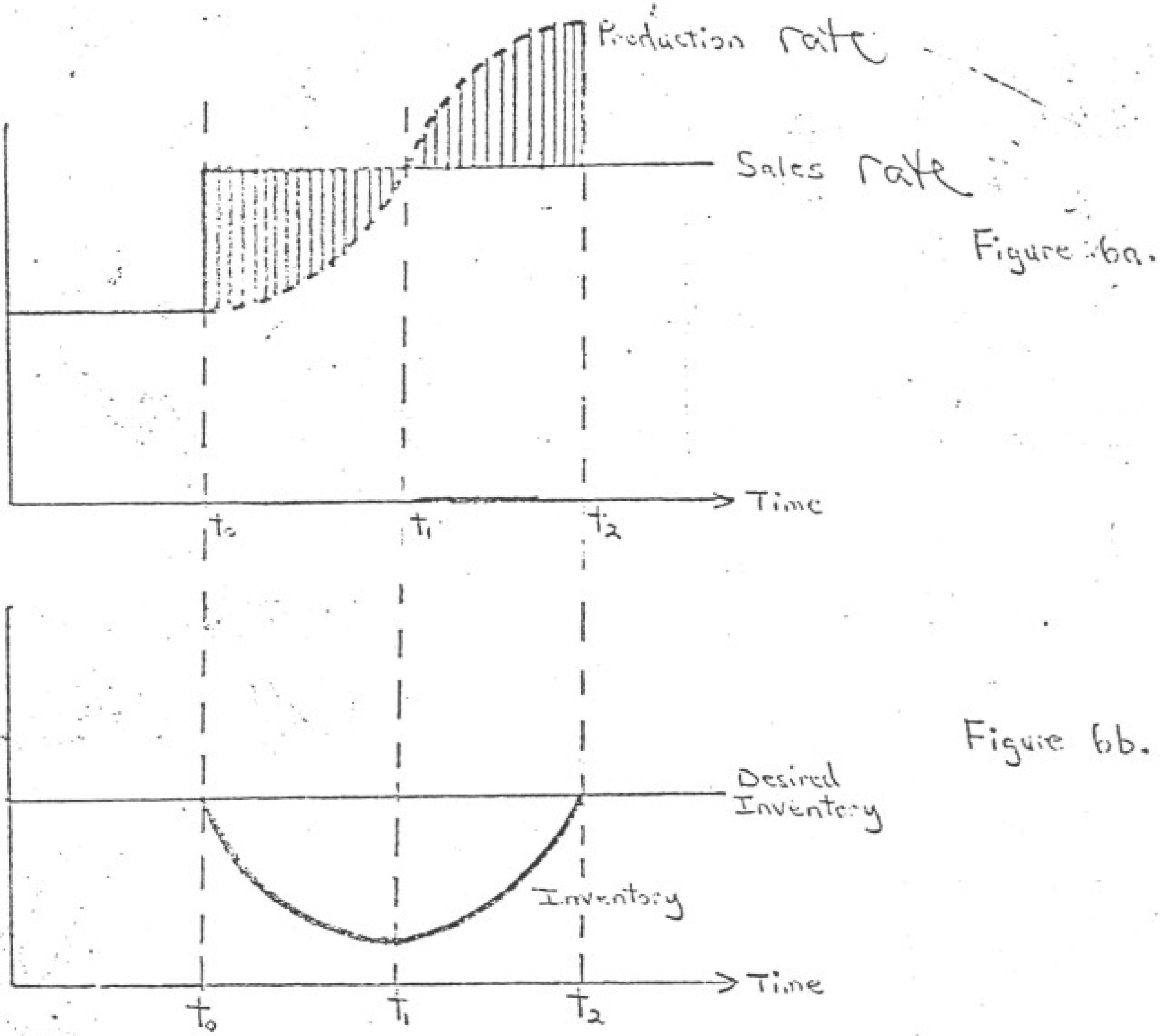


Figure 6. Rising Inventory as Production Exceeds Sales

At time t_2, inventory equals desired inventory but production rate is greater than sales rate. Production rate must rise above sales rate in order to regain the inventory lost between t_0 and t_1. In fact, for inventory to equal desired inventory, production must increase sufficiently above sales so that, in Figure 6a, the shaded area between t_1 and t_2 (representing the gain in inventory over this interval) equals the shaded area between t_0 and t_1. After time t_2, then, inventory continues to rise beyond desired inventory because production exceeds sales. As inventory goes above desired inventory, an inventory surplus is created. The surplus of inventory forces the hire-fire rate to become negative, representing net firing of employees, and workforce and production rate begin to decline. Workforce and production rate will continue to decline until inventory once again equals desired inventory.

By examining the behavior of production up until time t_3, when production rate once again equals sales, we can show that the production curve from t_2 to t_3 must, in fact be symmetric to the curve from t_1 to t_2. Figure 7 shows this symmetric curve, curve (1), and two alternative curves, curves (2) and (3), which begin to deviate from curve (1) at time t_2.

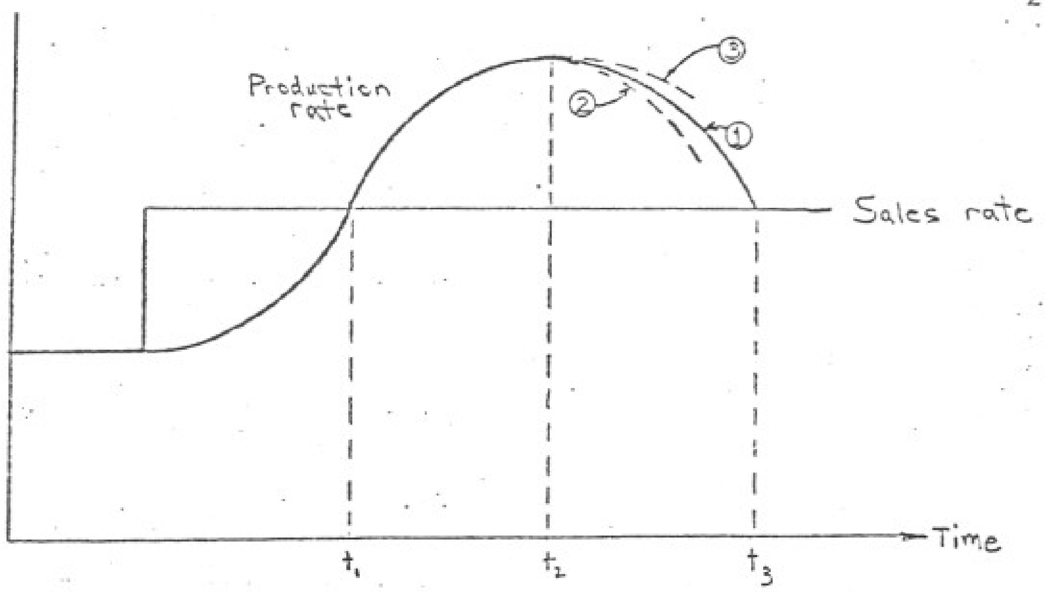


Figure 7. Alternative Paths for Production Rate Between t_2 and t_3.

In order to demonstrate that curve (1) is the proper path for the production behavior between t_2 and t_3 we first show that the symmetric production curve relates the upswing and downswing in production rate in a manner consistent with the model structure. According to the model, the slope of the production curve is determined by the rate of change of the workforce, that is, by the hire-fire rate; equation III.3 implies that the change in production rate over any time interval equals the change in workforce over that same interval multiplied by the constant productivity per man PPM.[4] The hire-fire rate (equation III.8), in turn, depends only on the discrepancy between desired inventory and inventory. Therefore, the net hiring rate corresponding to an inventory shortage of x units must equal the net firing rate corresponding to an inventory surplus of x units.

Curve (1), describing the production behavior between t_2 and t_3, meets the above condition relating the ascending and descending production curves. To see this, consider points t' and t", as shown in Figure 8, at which the production rate is the same:

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4. In analytic terms, equations III.3, III.6, and III.8, can be combined to

$$\frac{d(PR)}{dt} = (DINV - INV)/(IAT * WFAT)$$

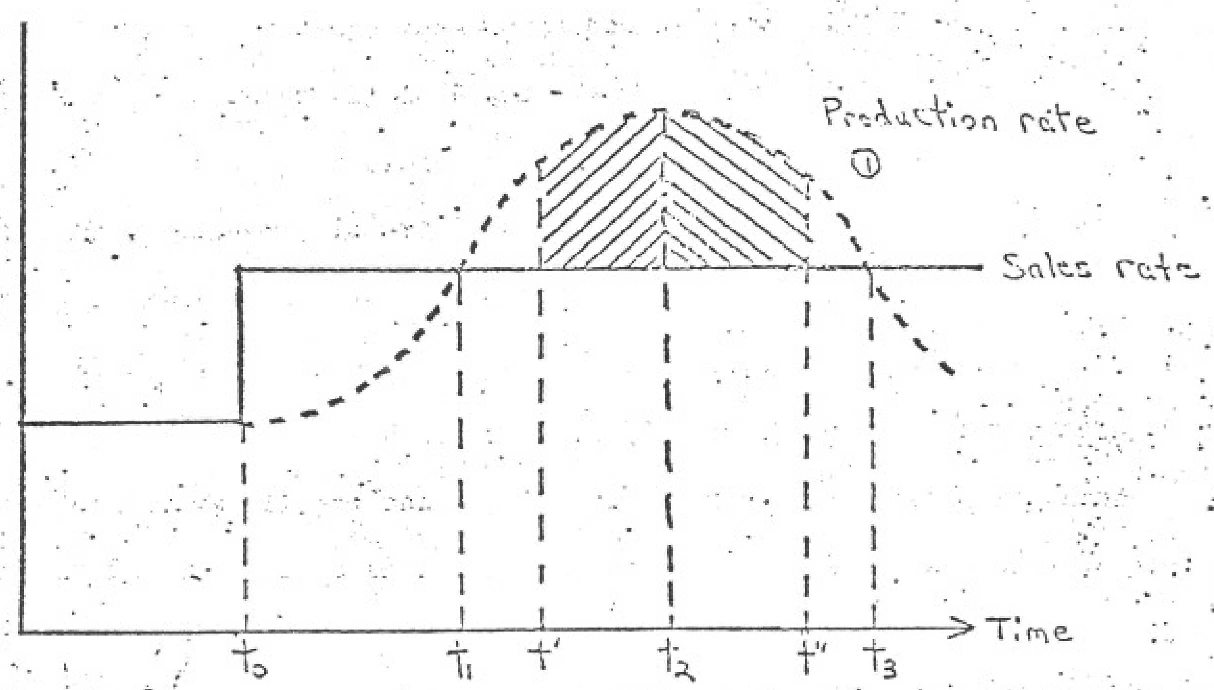

Figure 8. Demonstration That Production Curve (1) is Consistent with the Inventory-Workforce Model Structure.

Since curve (1) is symmetric to the production curve from t_1 to t_2, the slope of curve (1) at t" must be equal in absolute value to the slope t'. To be consistent with the model structure, then, the hire-fire rate at t" must be the negative of the hire-fire rate at t'. We can show this to be true by demonstrating that the inventory shortage existing at t' equals the inventory surplus at t". The inventory shortage at time t' is just equal to the shaded area between t' and t_2 (since inventory equals desired inventory at t_2). Analagously, the inventory surplus at time t" is just the shaded area between t_2 and t". Because of symmetry, these two areas must be equal; so HFR at t" will, in fact, be the negative of HFR at time t'.

Moreover, curve (1) is the only possible path between t_2 and t_3 that is consistent with the system structure. If production rate declined from its peak at t_2 more rapidly than described by curve (1), it would be described by a path such as curve (2) in Figure 7. If curve (2) falls off more rapidly than curve (1), the net firing rate must be greater than that corresponding to curve (1). However, in order for the net firing rate to be greater, the inventory surplus generated by curve (2) must be greater than that generated by curve (1). Yet, inspection of Figure 7 shows just the opposite to be the case: the area between production curve (2) and the sales rate is <u>less than</u> that between curve (1) and sales. Likewise, curve (3) must be rejected because it falls off less rapidly than curve (1) implying a smaller employee firing rate. However, curve (3) generates a larger inventory surplus which should necessitate a greater firing rate; curve (3), therefore, cannot describe the real system behavior. This same argument can be applied to show that the path of declining production cannot diverge from the symmetric curve (1) at any point in time between t_2 and t_3.

The symmetry between the rise of production between t_1 and t_2 and the fall of production between t_2 and t_3 guarantees that the areas under these two curves are equal. In turn, the equality of the two areas guarantees that the inventory oscillations in the model are undamped. Figure 9 illustrates this conclusion. Figure 9a divides the area between the production and sales rates into three sections. Figure 9b graphs the corresponding behavior of inventory over these three time segments.

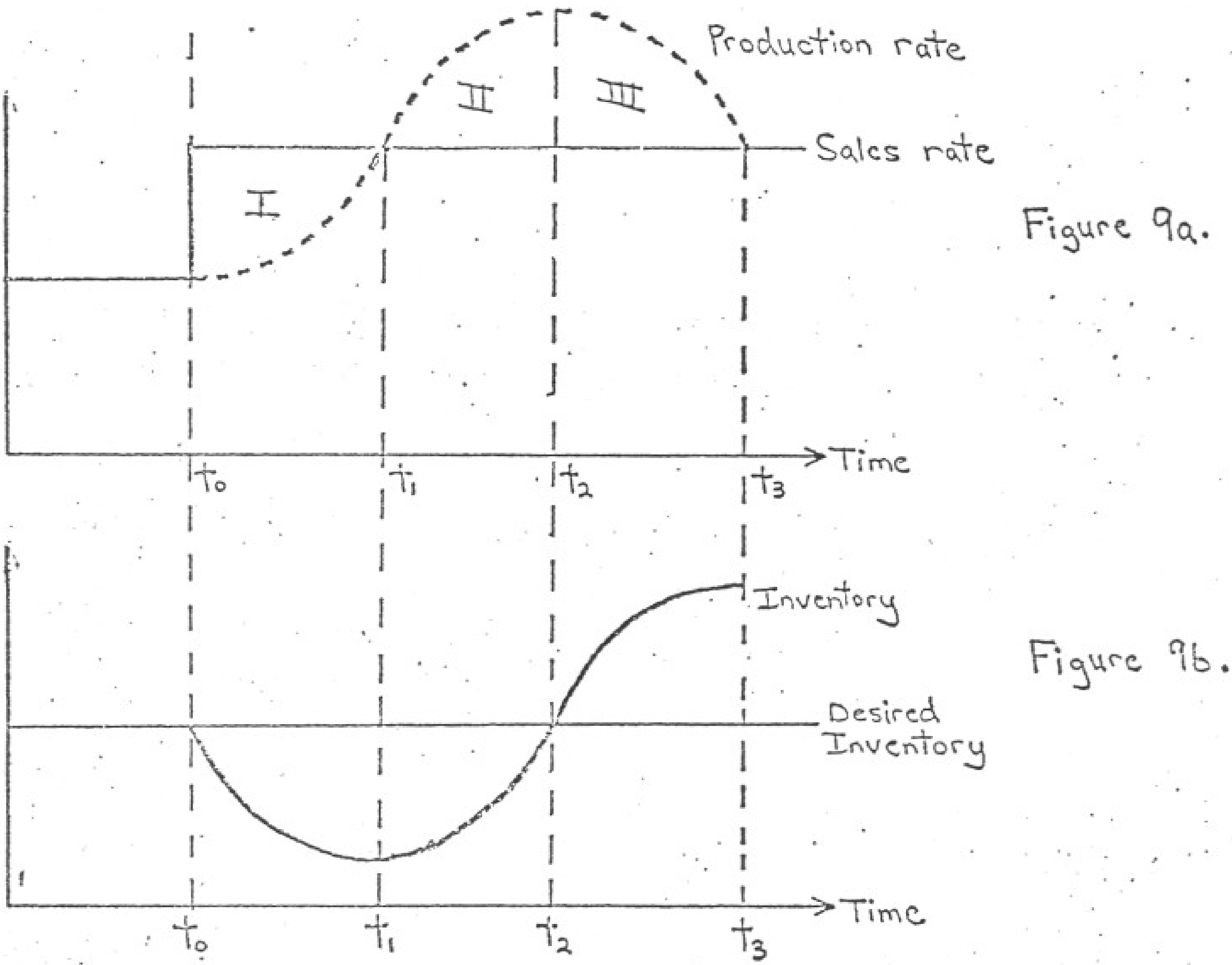

Figure 9. Undamped Oscillations in Inventory

Figure 9b shows that inventory continues to rise above desired inventory after t_2 until it reaches a peak value at t_3. In particular, inventory must rise above desired inventory by an amount equal to the inventory discrepancy at t_1. The values of inventory at the peaks and troughs depend on the areas between the production and sales curves, that is, on the amount of inventory gained or lost before production catches up with sales. The inventory oscillation in Figure 9b is undamped because area I equals area II in Figure 9a. Area III, in turn, equals area I because both areas equal area II.

The above analysis completes the demonstration that the second-order inventory-workforce model can only exhibit undamped oscillations. Figure 10 is an actual computer simulation of the inventory-workforce system. Looking at the production curve over time in Figure 10, we can now see that area I equals area II, which equals area III, which equals area IV, and so on. The next two sections of the paper will employ a similar method of graphical analysis to illustrate the effect of two structural changes on the oscillatory behavior of inventory and production rate.

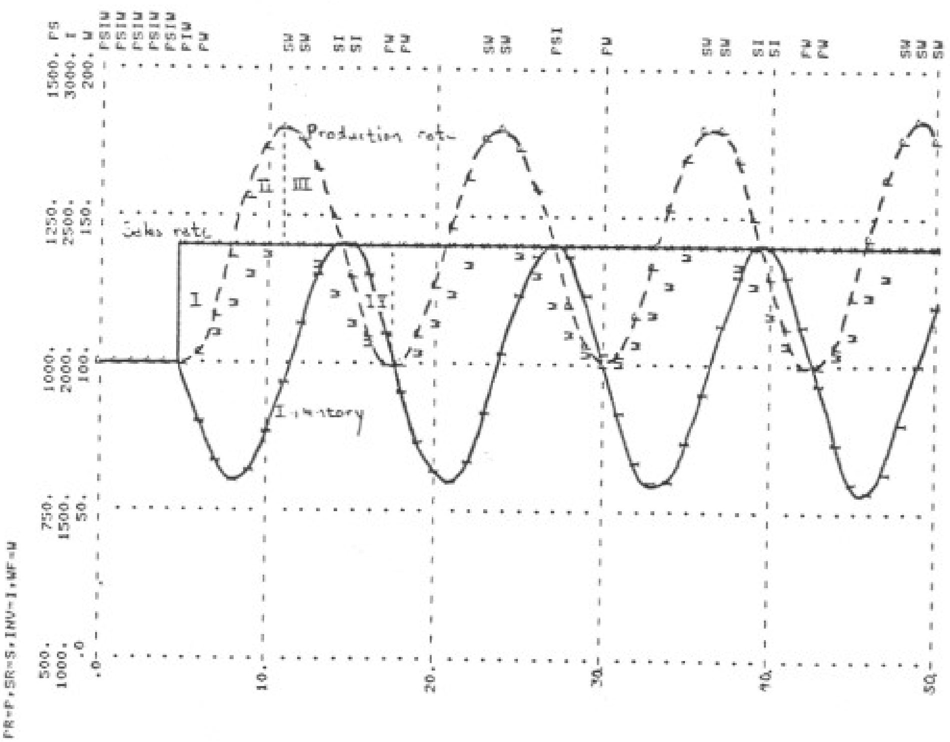

Figure 10. Computer Simulation
of Undamped Inventory-Workforce System

IV. EXPANDING OSCILLATIONS IN A THIRD-ORDER SYSTEM

The inventory-workforce model presented in Section III assumed that there is no delay between the decision to produce output and the time that output is ready to be shipped--production starts were identical to final output. By addition of a flow delay between the decision to produce inventory and the arrival of that inventory, the inventory-workforce system can be expanded to contain an explicit level of in-process inventories. Section IV extends the method of graphical analysis presented in the preceding section to show why addition of such flow delay can cause inventory and workforce oscillations to expand over time.

The structural modification explored in this section is presented in Figure 11. Production rate PR now flows into a level of in-process inventory IPI. Finished production FP flows out of IPI at a rate determined by the time constant, inventory-production delay IPD. The additional equations modifying the original inventory-workforce model are given below Figure 11.

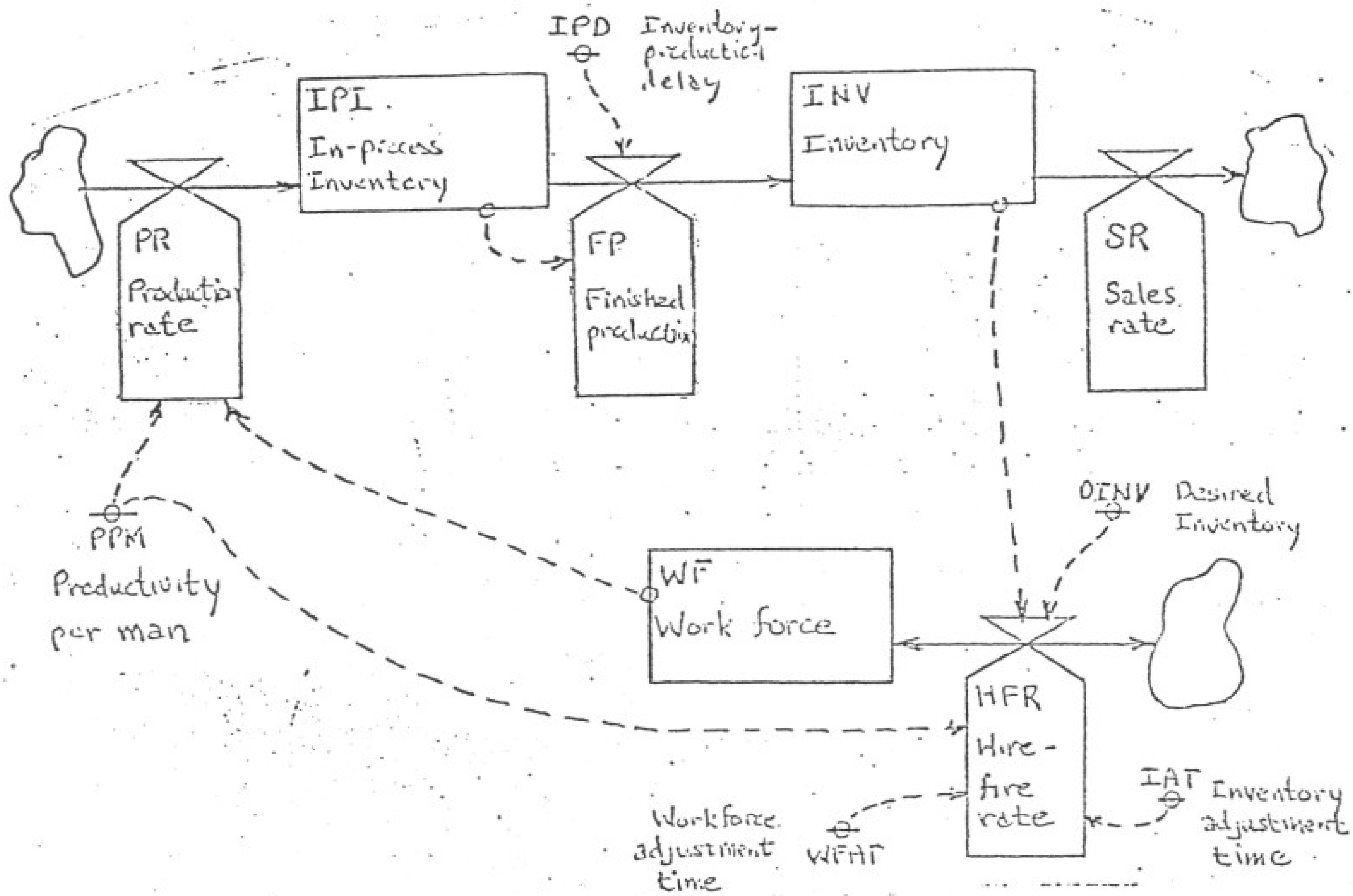

Figure 11. Inventory-Workforce Model
with Inventory-Production Delay

IV.1	L	INV.K = INV.J + (DT)(FP.JK - SR.JK)	Inventory (units)
IV.2	R	FP.KL = IPI.K/IPD	Finished production (units/month)
IV.3	C	IPD =	Inventory-production delay (months)
IV.4	L	IPI.K = IPI.J + (DT)(PR.JK - FP.JK)	In-process inventory (units)
IV.5	N	IPI = PR * IPD	

As a result of the inventory-production delay, inventory shortages produced by an exogenous sales increase continue to build up over a longer period of time than previously. The initial imbalance of sales and production following the step up in sales rate causes inventory to be depleted and, in turn, stimulates an increase in workforce and consequently a rise in production rate. However, finished production FP lags production starts; therefore, inventory shortages continue to increase, beyond the point when production equals sales (time t_1), until the time when finished production equals sales (time t_1'). Figure 12 shows the resulting inventory shortage as the shaded area I between the FP curve and sales rate from t_0 to time t_1'.

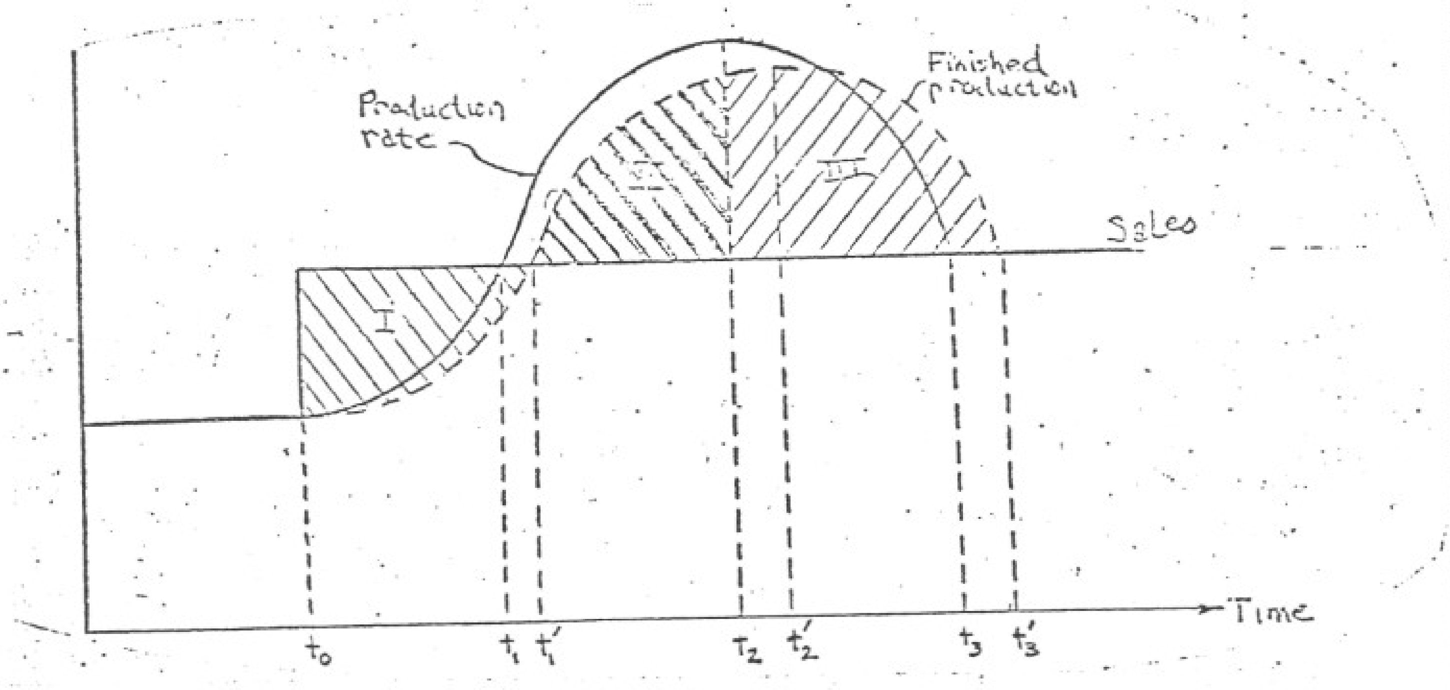

Figure 12. Behavior of Inventory-Workforce System with Inventory-Production Delay

Analysis of system behavior between time t_1' and t_2 shows that oscillations in production rate PR are expanding--that is, PR moves progressively further from sales on each successive swing. The production rate curve rises until the entire inventory shortage built up between t_0

and t_1' is made up. The inventory shortage is finally eliminated at time t_2, when the shaded area II in Figure 11 (between the FP curve and sales) equals area I. The equality of areas I and II implies that the distance between finished production and sales at time t_2, $FP(t_2)-SR(t_2)$, approximately equals the distance between FP and SR at time t_0.[5] However, due to the inventory-production delay, the PR curve is above the FP curve at t_2. Therefore, the production rate peak at time t_2 is further from sales than was the original production trough at time t_0. Subsequent swings in production continue to move farther and farther from sales. Moreover, expanding oscillations in production mean that oscillations in workforce are also expanding over time (since production equals a constant, PPM, times workforce).

Inventory oscillations are also expanding, as can be seen from the area between the FP and sales curves in Figure 12. Finished production continues to increase beyond the point when inventory equals desired inventory (time t_2). As a result of the flow delay, finished production continues to increase until time t_2' and only intersects the sales rate at time t_3'. The value of the initial trough in inventory, which occurred at t_1', was equal to area I. The inventory surplus that accumulates up until time t_3' equals the area between FP and SR from time t_2 to time t_3' (area III). Comparison of areas I and III shows that, as a consequence of the inventory-production delay, the surplus of inventory that builds up while FP is above sales exceeds the shortage of inventory accumulated while FP was below sales. That is, the oscillations in inventory are expanding.

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5. In fact, the distance between $FP(t_2)$ and $SP(t_2)$ is slightly greater than the distance between $FP(t_0)$ and $SP(t_0)$. This occurs because the slope of the ascending FP curve continues to increase for a short time after t_1', causing the initial inventory shortage to be eliminated over a somewhat shorter interval than $t_1'-t_0$ (i.e., $(t_2-t_1) < (t_1-t_0)$. But, area I equals area II. Therefore, $FP(t_2)$ must be farther from sales than is $FP(t_0)$. This guarantees that $PR(t_2)$, which exceeds $FP(t_2)$, must likewise be farther from sales than is $PR(t_0)$.

Repetition of the preceding analysis to cover the succeeding fall and rise in production rate will reveal a continuing expansion in the amplitude of oscillations for the inventory-workforce system. The next trough in production will be further from sales than was the peak in production at time t_2. The subsequent swing in finished production will produce an inventory shortage which will exceed the surplus built up between time t_2 and t_3'. In this way, the inventory-production delay will produce continuously expanding swings in production and inventory.

V. A POLICY FOR STABILIZING PRODUCTION AND INVENTORY FLUCTUATIONS

Section III showed how undamped oscillations were produced in the inventory-workforce system when the hire-fire rate HFR was assumed to depend on the discrepancy between actual and desired inventory. As analyzed in section III, the resulting system had several characteristics which contributed to inventory and production fluctuations. According to the hire-fire rate policy, workforce was continually expanded whenever there was an inventory shortage and contracted whenever an inventory surplus was present. These adjustments were made independent of whether workforce was initially above or below the level required for production to cover average sales rate. Thus, for example, production rate was continually increased above the sales rate in the face of inadequate inventory, ignoring the fact that production rate would be excessive once the inventory shortage were eliminated. As a result, the peak of production occurred at the point when inventory just equalled desired inventory. This, in turn, implied that the rate of inventory expansion was greatest when inventory equalled desired inventory. Such behavior is clearly undesirable for the typical firm.

In light of the comments above, we might consider reformulating the equation for HFR to subsume both an inventory-correction term and an adjustment of workforce to achieve the production rate required to cover the long-term sales. The revised formulation for HFR appears in Figure 13. In equation V.2, the desired workforce DWF is the sum of the workforce desired for sales WDS and workforce desired for inventory production WDIP. As shown in equation V.3, WDS is simply the sales rate divided by the productivity per man PPM.[6] In equation V.4 for WDIP, the inventory discrepancy, (DINV-INV), divided by the inventory adjustment time IAT yields the desired inventory production rate. Dividing the desired inventory production rate in turn, by PPM, yields the number of men needed

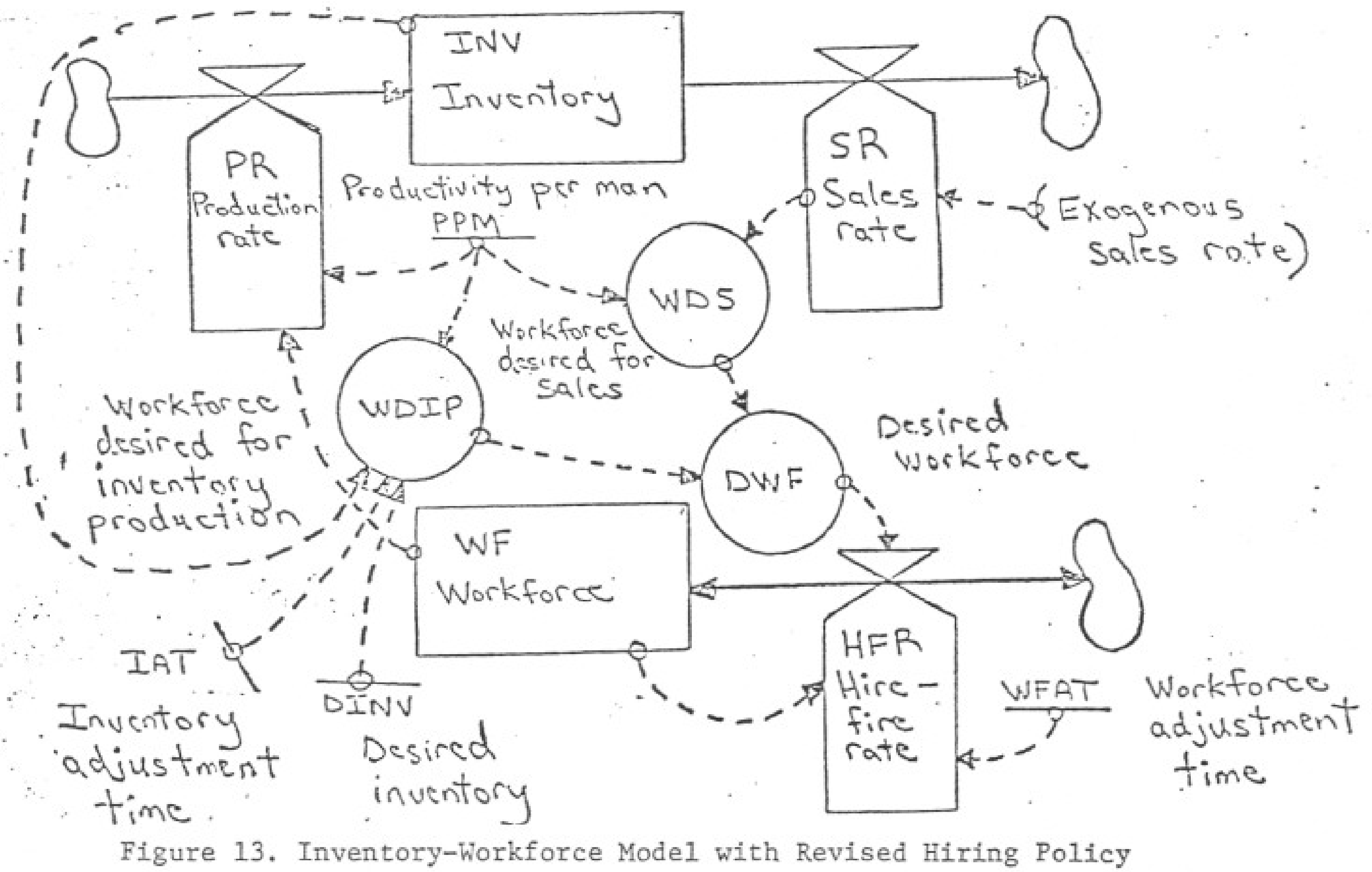

Figure 13. Inventory-Workforce Model with Revised Hiring Policy

6. As a simplification, WDS is assumed to depend on the actual sales rate, rather than on an average sales rate as would be perceived by a firm's management. This simplification avoids the additional complexity of dealing with a third-order system in the explanation below. Moreover, in this example, omission of the averaging process has little effect on system behavior due to the presence of a constant sales rate.

for inventory production. The hire-fire rate HFR is simply desired work-force minus workforce, divided by the workforce adjustment time WFAT.

V.1	R	HFR.KL = (DWF.K - WF.K)/WFAT	Hire-fire rate (men/month)
V.2	A	DWF.K = WDS.K + WDIP.K	Desired workforce (men)
V.3	A	WDS.K = SR.JK/PPM	Workforce desired for sales (men)
V.4	A	WDIP.K = (DINV - INV.K)/(IAT * PPM)	Workforce desired for inventory production (men)

Using the method of graphical analysis demonstrated in sections III and IV, we may now see why the new hiring policy produces damped (convergent) oscillations in production and inventory. Writing the equation for hire-fire rate HFR as a single expression, we obtain:

$$HFR = \left(\left(\frac{SR}{PPM}\right) + \left(\frac{DINV - INV}{IAT * PPM}\right) - WF\right) \Big/ WFAT$$

The second term of the equation, (DINV - INV)/(IAT * PPM * WFAT), is identical to the earlier formulation of HFR (see equation III.8). Added to this now is a second term

((SR/PPM) - WF)/WFAT.

This term will be positive, producing net hires, when the sales rate exceeds production rate (since SR > PR implies SR > WF * PPM; that is, SR/PPM > WF). Conversely, the term will be negative, producing net firing of workers, when production rate is greater than sales rate.

Thus, when sales rate is increased at time t_0, production rate will expand more rapidly than it did under the old policy. Rapid hiring is caused by the positive inventory discrepancy (DINV-INV) and by the fact that production is now less than sales. As a result of higher production, the inventory discrepancy in Figure 14 between t_0 and t_1 will be smaller than the corresponding discrepancy under the old policy. At time t_1, production equals sales.[7] But there remains a positive inventory discrepancy, so net hiring will continue, raising production above sales. Once production exceeds sales, downward pressure is exerted on the hire-fire rate HFR. However, at this point inventory is still below desired inventory, thereby exerting upward pressure on hires. At first, when production is only slightly above sales, the inventory effect will dominate, causing hiring and production to expand. But as production continues to rise, pressures will mount under the new policy to cut back employment. As a result, production rate will peak and decline <u>before the inventory discrepancy has been fully eliminated</u>. In Figure 14, then, the shaded area between t_0 and t_1 will exceed the area from t_1 to t_2.

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7. Note that the time required for production to rise up to sales will be less under the new policy than with the old policy. Nonetheless, for consistency, we retain the nomenclature t_1, t_2, and so on, to indicate successive peaks and troughs in production, or points at which production equals sales.

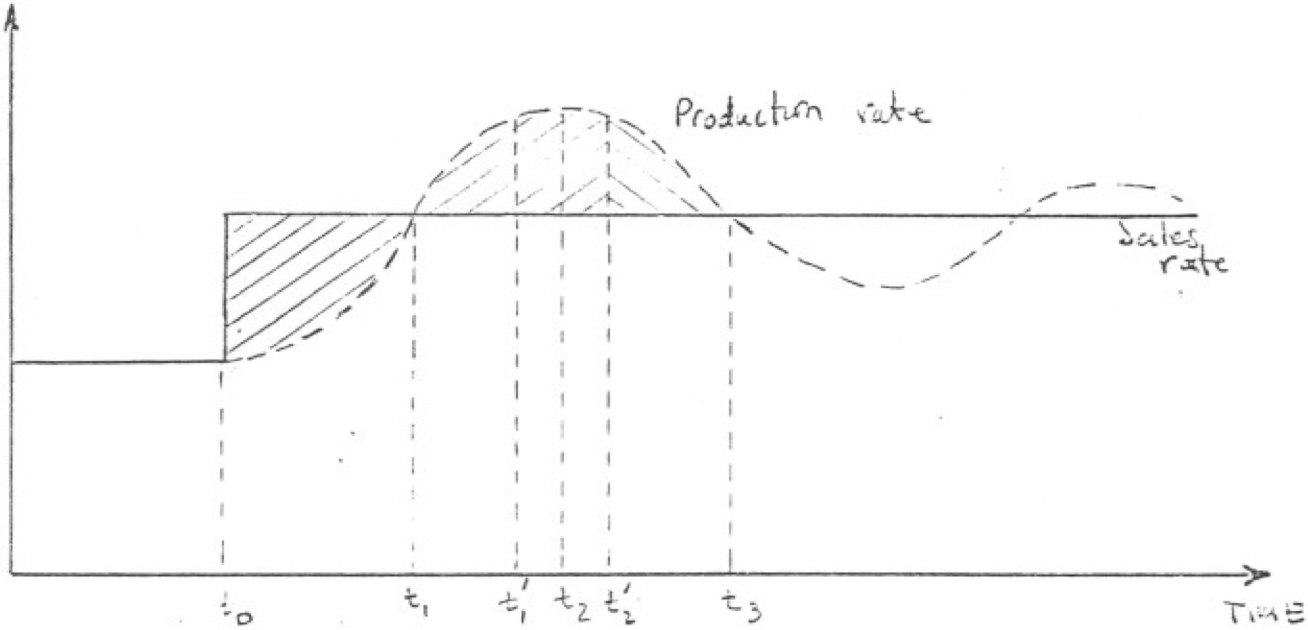

Figure 14. Behavior of Inventory-Workforce System with Revised Hiring Policy

Continuing the analysis up to time t_3, when production rate again equals sales rate, we can see that there must be some point between t_2 and t_3 where the initial inventory shortage has been made up and inventory INV once again equals desired inventory DINV. Call this point where INV = DINV t_2'. Only after time t_2' does an inventory excess begin to accumulate. As can be seen from Figure 14, the inventory excess which accumulates between t_2' and t_3 is considerably smaller than the inventory shortage that accumulated between t_0 and t_1.

Repetition of the above analysis for each successive swing in production rate shows that the corresponding swings in inventory contract over time.* Thus, the behavior of production and inventory over time must be damped oscillations towards the sales rate and desired inventory, respectively. It may also be seen intuitively that the damping in the system will increase--that is, the degree of overshoot in production will decrease and the system will near equilibrium after fewer oscillations--as the inventory adjustment time IAT is lengthened. Consider a point such as t_1' in Figure 14. At t_1', production exceeds sales, exerting downward pressure on hiring, but inventory is below desired inventory. As IAT becomes larger, the relative importance of the inventory discrepancy as it affects hiring is diminished. Thus, in the equation for HFR, the term depending on production rate minus sales rate will tend to dominate, lowering production and causing the production peak to occur earlier and at a lower value. If a peak in production is characterized by a larger inventory shortage than was the case for shorter IAT, there will be a smaller inventory surplus than previously when production once again equals sales at time t_3. The size of the inventory surplus at time t_3 determine how far production rate will have to fall below sales rate before the firm will stop firing workforce

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*A formal proof of the proposition that the area under the production rate curve between t_2' and t_3 is less than the area between t_0 and t_1 requires algebraic manipulations of the equation describing production rate as a function of time and therefore is not included in the present paper. The above verbal analysis, however, contains the essential points needed to grasp why the revised hiring policy produces damped oscillations in inventory.

applying the graphical analysis methods used in the preceding discussions to the case of lengthening IAT (or diminishing PPM).

VI. CONCLUSIONS

Successful application of system dynamics requires that the model builder be able to communicate to potential users why a particular model behaves as it does. Such understanding is needed to establish confidence in a model and is a necessary prelude to the use of a model for policy design. Yet, model-builders and users often do not strive for penetrating descriptions of how underlying causal mechanisms produce observed behavior.

This paper has employed graphical analysis techniques to provide insight into the mechanisms underlying behavior of some simple oscillatory systems. These techniques should be equally helpful in analyzing other relatively simple systems. Moreover, as typical classes of simple structures are isolated and their behavior understood, the resulting insights should provide modelers with a basis for understanding the behavior of more complex oscillatory systems.