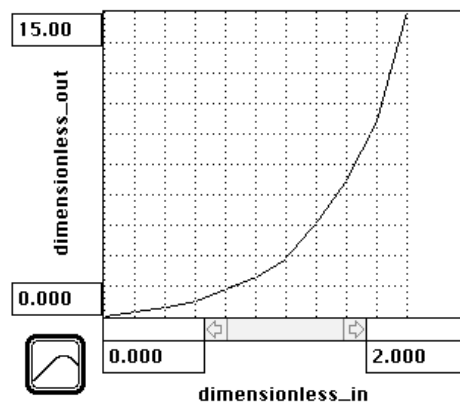# Mistakes and Misunderstandings:

Table Functions

Prepared for the
MIT System Dynamics in Education Project
Under the Supervision of
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by
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July 15, 1997
Vensim Examples added October 2001

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1. THE SYSTEM

"A population model? Oh sure, I know how to make population models. They're all over Road Maps..."

I sat down at my desk and stared at the information I had gathered to build my model. Aardvarks. Stout, pig-like animals up to six feet long (hmm, longer than I am tall) with a long snout, rabbit-like ears, and short legs. Coat varies from glossy black and full to sandy yellow and scant. The aardvark is native to Africa. Today approximately one million aardvarks roam the savannah south of the Sahara. The name aardvark is Afrikaans for "earth pig."

Figure 1: An aardvark

My population model needed to describe the spread of the aardvark population. A few aardvarks searching for food discover a lush savannah. They settle down and, as seasons pass, reproduce and grow in number. Typically, a young aardvark is born to each middle-aged female aardvark once a year. After several years, though, the savannah is no longer a paradise for aardvarks. Many aardvarks compete for the same food, the same shelters. Due to limited natural resources, aardvarks die at a faster rate. Eventually, the population in the savannah stabilizes.

I realized that the key to my population model would lie in correctly representing the constraint of the environment on the aardvark system. The death fraction is a

nonlinear function of the aardvark population. A small increase in population for a population that is small (when compared to the maximum population the environment can support) only slightly changes the death fraction. A small increase in population for a population that is large (with respect to the maximum population), however, causes a large increase in the death fraction. Therefore the relationship between population and death fraction is nonlinear. Table functions are excellent tools for modeling nonlinear relations.

Table functions are graphical relationships, usually nonlinear, between two variables. Table functions graph a table of data points. They allow a modeler to easily specify relationships that are not amenable to algebraic description. Table functions often aggregate several real processes into one relationship. In general, table functions are used for the representation of nonlinear variables. The effect of the constraint of the environment, which is negligible when relatively few aardvarks live in the savannah and overwhelming when many aardvarks roam around, is a nonlinear variable.

2. A FIRST ATTEMPT TO MODEL THE SYSTEM

In my first attempt to model the aardvark system, I created the model in Figure 2. Aardvarks are the total number of aardvarks currently living in the savannah. The number of aardvarks is increased by births and decreased by deaths. In the model the birth fraction is constant while the death fraction changes with the current number of aardvarks.
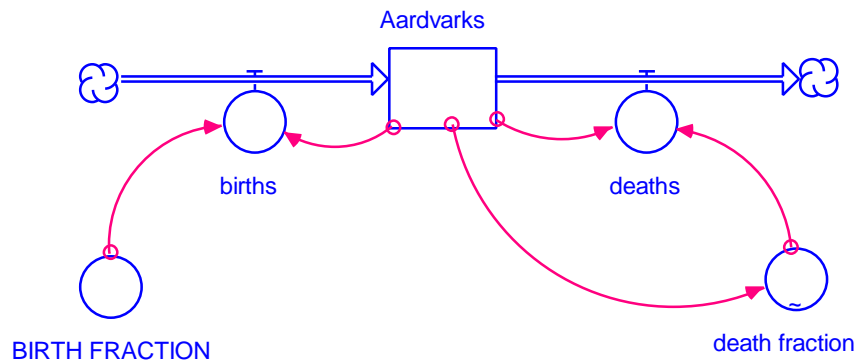

Figure 2: My first aardvark population model

Female aardvarks give birth to one baby each year. Estimating that about half of the aardvark population is female, and more than half of those females are able to reproduce (not too young, not too old) then approximately one third of the aardvarks are able to give birth each year. So I entered an approximate "BIRTH FRACTION" of 0.3 aardvarks/aardvark/year (which is equivalent to 0.3 per year).

I then tackled the "death fraction". I plotted the "death fraction" as a function of the current number of "Aardvarks" in the table function shown in Figure 3. My first step in obtaining the curve was to identify the values of the "death fraction" for its extreme cases, the limits of the curve. I know that aardvarks have an average lifespan of 10 years. So initially the "death fraction" is 0.1 per year.[1] Very large aardvark populations, however, are constrained by the environment. I referred to an encyclopedia to find out that, at most, 500 aardvarks live in a 1000 acre savannah. So when a 1000 acre savannah holds 500 aardvarks, the population should be at equilibrium... which means that the number of aardvark deaths must equal the number of births. Therefore, when the population of aardvarks reaches 500 animals, the "death fraction" should equal the "BIRTH FRACTION", 0.3 per year. My next step in obtaining the curve was to estimate the overall shape of the curve connecting the extreme points. The curve between 0.1 and 0.3, between the two extremes, increases slowly for relatively small populations when natural resources abound and then increases sharply for large populations as the natural resources become scarce.

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[1] The death fraction is the reciprocal of the time constant. Here, the time constant is the average lifespan of an aardvark, the time for an aardvark to die, 10 years.
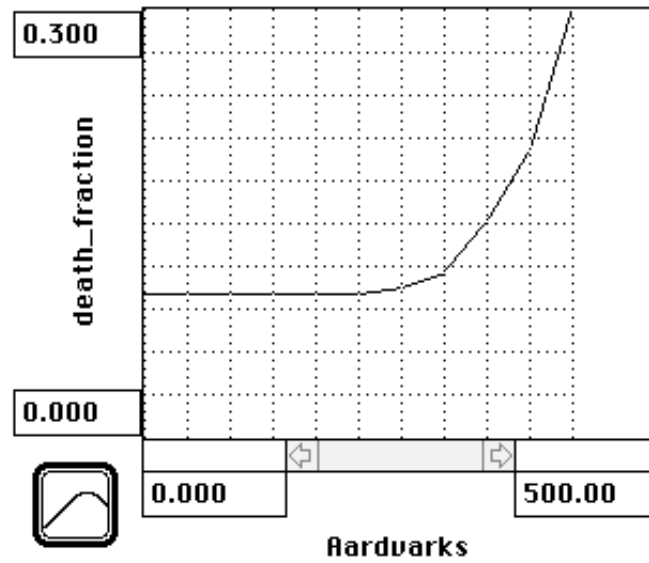
Figure 3: The table function for my first model

With my table function complete, I ran the model and analyzed the graphs I obtained. Figure 4 depicts the behavior of the aardvark population model: S-shaped growth. Proud of my modeling skills, I submitted my work.
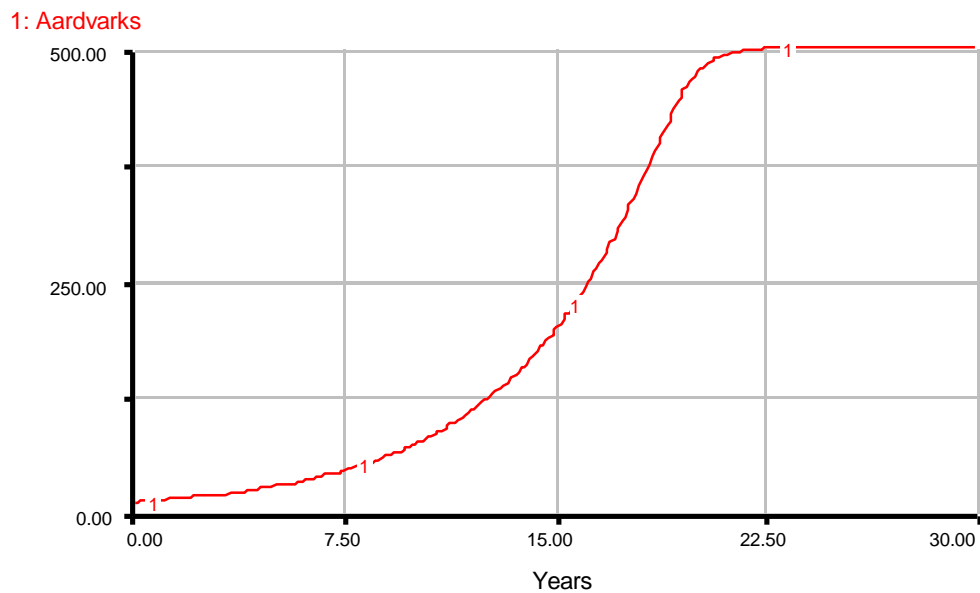

Figure 4: Behavior of the aardvark population

3. FIRST MISTAKE AND MISUNDERSTANDING

One week later the professor returned our papers. I eagerly flipped the paper over and my jaw dropped as I couldn't even recognize my work, buried beneath a bloody trail of red ink. I quickly hid it in my notebook and did not look at it again until late that afternoon, when I had reached the privacy of my own room. Bold letters along the front page blared "inputs and outputs to a table function must be dimensionless." I did not see why dimensionless inputs and outputs were such a big deal. My model produced the right type of behavior—so where was the problem? Convinced that my model had been unfairly criticized I took it over to my friend Mathilda's house to complain about the injustices of this world. Mathilda looked at my model and then looked back up at me, surprised. She asked me to run through the loops and explain what my units were. I showed her that the "births" flow is the product of the number of "Aardvarks" and a fraction per year, which combine to give a flow of aardvarks/year. Likewise, the "deaths" flow is the product of the number of "Aardvarks" and another fraction per year, which also combine to give a flow of aardvarks/year. Then I suddenly realized my mistake. The "death fraction" as I had modeled it was not in units of 1/year but in units of aardvarks, because the current number of aardvarks drives the table function. I wanted my table function to output units of 1/year, but I had given it an input of aardvarks. Table functions do not mysteriously change units; whatever units go into the table function will come right out. I groaned; I had submitted a model that violated dimensional consistency. I started to feel that I might have deserved all the red editorial ink.

4. A SECOND ATTEMPT TO MODEL THE SYSTEM

Mathilda went to get her laptop and we sat down at her desk to look at my model. She explained that my model needed normals and multipliers. A normal is a reference point that is not changed by the behavior of the system. My model implicitly contained two reference points. I had hidden two critical parameters in my table function: a maximum aardvark population and an initial death fraction. The maximum population of

aardvarks in a 1000 acre savannah, a "normal" number of aardvarks, is 500 aardvarks. The maximum population works as a constraint on the system representing the limited natural resources. I decided to call this ratio of "Aardvarks" to the maximum aardvark population "fraction occupied." The variable "fraction occupied" denotes what fraction of land and natural resources are currently taken up by aardvarks. The extent of crowding in a savannah does not depend on an *absolute* number of aardvarks but depends on a *relative* number of aardvarksóthe current number of aardvarks compared to the maximum number that the natural resources of the savannah are able to sustain. The "normal" death fraction is the death fraction that would occur under "normal" conditions: when the aardvark population grows freely without being constrained by the environment. The "normal" death fraction is 0.1 per year. As the environmental conditions change, the death fraction changes relative to the fraction of aardvarks that would die under "normal" conditions.

Mathilda and I decided to add the normal values to my model. We drew up a new model of the aardvark population that is depicted in Figure 5.
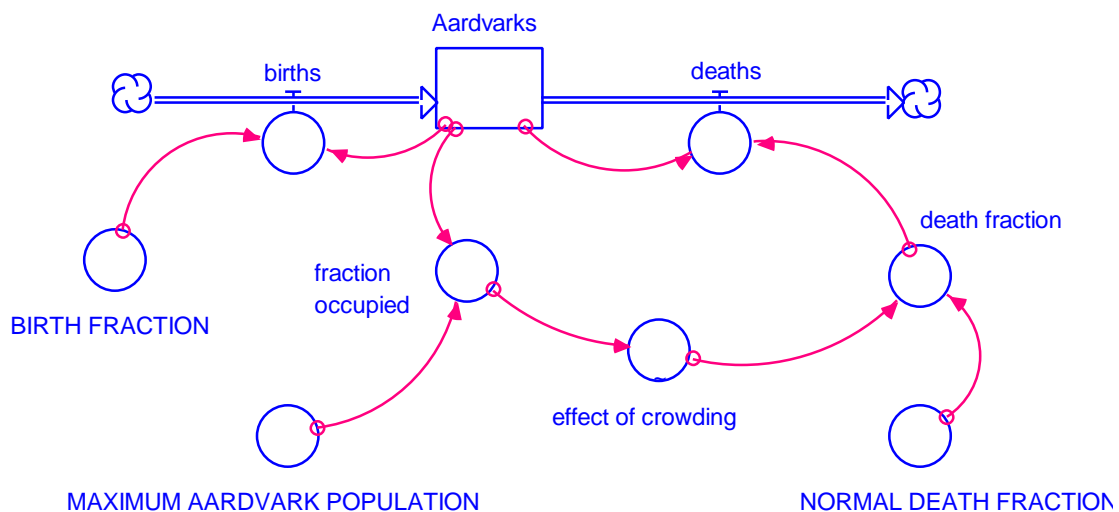

Figure 5: My second model with normals and a multiplier

In the new model both of the normal values are explicit: they have been pulled outside of the table function. The normal values are constants; they do not change as the aardvark population evolves. The changing effect of the number of aardvarks on the death

rate is captured by a multiplier: the "effect of crowding." The "effect of crowding" is a multiplier because it multiplies the "NORMAL DEATH FRACTION" with a weight that represents the changing conditions of the system, the fraction of land and resources occupied by the aardvarks. The "fraction occupied" is calculated by taking the ratio of "Aardvarks" to the "MAXIMUM AARDVARK POPULATION." Insignificant at first, the "effect of crowding" multiplier increases as the fraction of land and resources occupied increases. The multiplier then combines with the "NORMAL DEATH FRACTION" to form a "death fraction" that increases as crowding becomes severe. Under normal conditions, when natural resources are abundant and the population can grow without being constrained by the environment, the "effect of crowding" multiplier must be 1.0 so that the "death fraction" equals the "NORMAL DEATH FRACTION." The curve of the table function in the "effect of crowding" provides the nonlinear relationship between the relative number of "Aardvarks" and the extent to which the "death fraction" will change beyond the "NORMAL DEATH FRACTION."

I drew the table function in Figure 6 by first considering how the behavior of the aardvark system changes under extreme conditions. When just a few "Aardvarks" live in the savannah, the fraction of land and resources occupied is minimal (the fraction is close to 0), and crowding is negligible. The multiplier must then be 1.0 so that the "death fraction" will equal the "NORMAL DEATH FRACTION." When the 1000 acre savannah is populated with 500 aardvarks, then the number of "Aardvarks" equals the "MAXIMUM AARDVARK POPULATION", the "fraction occupied" is 1.0, the "effect of crowding" is 3.0, and the system has reached its equilibrium. The "death fraction" equals three times the "NORMAL DEATH FRACTION," which equals the "BIRTH FRACTION" of 0.3 per year. In between the extremes, I know that the multiplier increases slowly when there are few aardvarks and increases drastically when the population is close to reaching its limit. Therefore the slope of the curve is small—almost flat—at first and then steepening as the population increases rapidly.
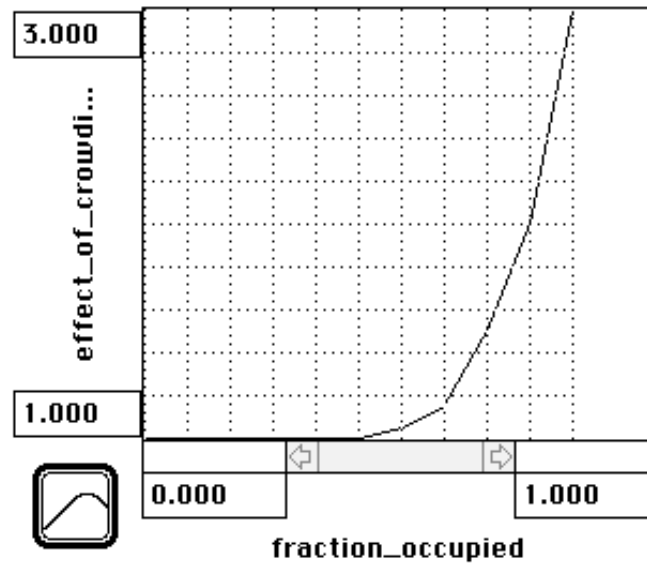
Figure 6: Table function with dimensionless inputs and outputs

Notice that in Figure 6 the input to the table function, the "fraction occupied," is the ratio of "Aardvarks" to the "MAXIMUM AARDVARK POPULATION," and is *dimensionless*. The input is not an absolute number of aardvarks, but the current number of aardvarks relative to a normal. The output is therefore also dimensionless. The output effect of crowding multiplies the "NORMAL DEATH FRACTION" (units of 1/year) with a dimensionless constant to form a "death fraction" that also has units of 1/year. The model is now dimensionally consistent. I ran the model and again obtained the same S-shaped growth as shown in Figure 4. I realized that reasonable behavior alone did not prove that I had a reasonable model. Sleepy, I printed out my model, slipped it into my backpack, and decided to ask my professor to please take another look at it.

5. SECOND MISTAKE AND MISUNDERSTANDING

A few days later my professor handed me back my model a second time. He smiled. "Well, this model is definitely much better than your first, but it still isnít robust," he said. I did not understand what he meant. "Your exercise for tonight," he explained, ìis to find out what it means for a model to be robust. I have one hint for you: try changing the birth fraction from 0.3 to 0.4 per year and think carefully about the results you obtain." After class I went straight over to Mathilda's house. We brought the model

up onto the screen and changed the "BIRTH FRACTION" from 0.3 to 0.4 per year. Figure 7 shows the results of our simulation.
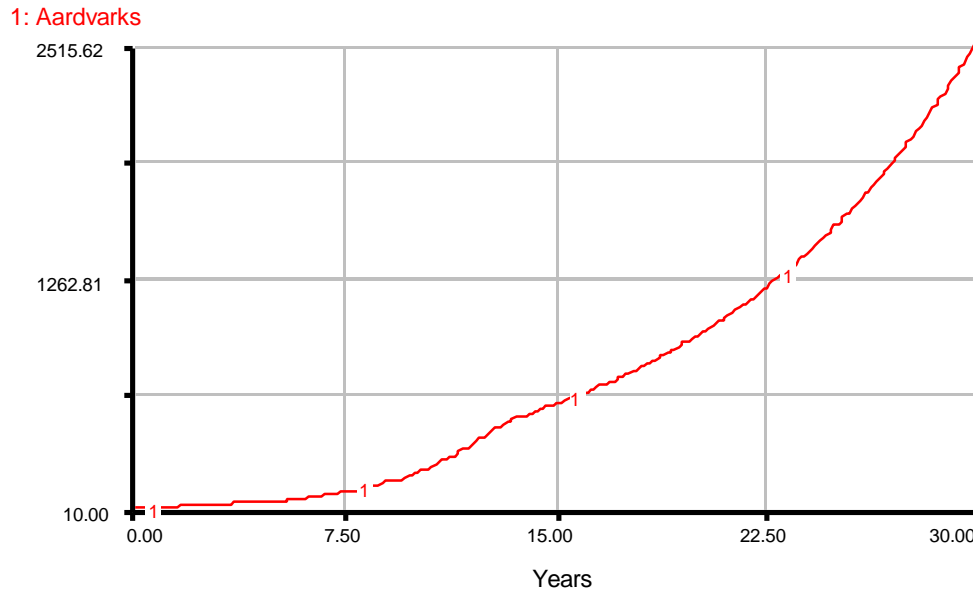

Figure 7: Behavior generated with a "BIRTH FRACTION" of 0.4 per year

6. OVERCOMING OUR MISTAKES AND MISUNDERSTANDINGS

Why did the model exhibit exponential growth? I was puzzled until I remembered that I had carefully crafted my table function to reach equilibrium when the "death fraction" caught up to the "BIRTH FRACTION," assuming a "BIRTH FRACTION" of 0.3 per year. When the number of aardvarks reaches the size of the maximum aardvark population, the "death fraction" reaches a maximum of 0.3 per year. But the "BIRTH FRACTION" of 0.4 is larger than the "death fraction" so the population continues to grow. The "death fraction" is limited by my choice of the endpoints of the table function, so it remains at 0.3 per year. Crowding therefore only slows down—but does not stop— the growth of the population. My awkward formulation of the table function, not the underlying dynamics of the system, generated an alternate, unrealistic form of behavior for the aardvark population. When my professor said that my model was not robust, he meant that the behavior that my model generated was not realistic for the entire range of

possible, realistic parameter values. The approximated aardvark birth fraction of 0.3 per year has a certain degree of uncertainty; an aardvark birth fraction could just as easily be 0.2 or 0.4 per year, depending on the specific system being modeled. My model was not robust because, although it worked fine for a "BIRTH FRACTION" of 0.3 per year, it broke down for a just-as-likely "BIRTH FRACTION" of 0.4 per year. My table function was only valid for very specific parametersóchange the parameters and my model goes from stabilizing the number of aardvarks to unrealistically growing the population exponentially. A strong dependence on a choice of parameters can mean different things. On one hand, I could have just discovered a critical leverage point within my system. On the other hand, my model couldn't and didn't simply suffer from not being robust.

I decided to expand my table function in order to make my model robust. Study the new table function in Figure 8.
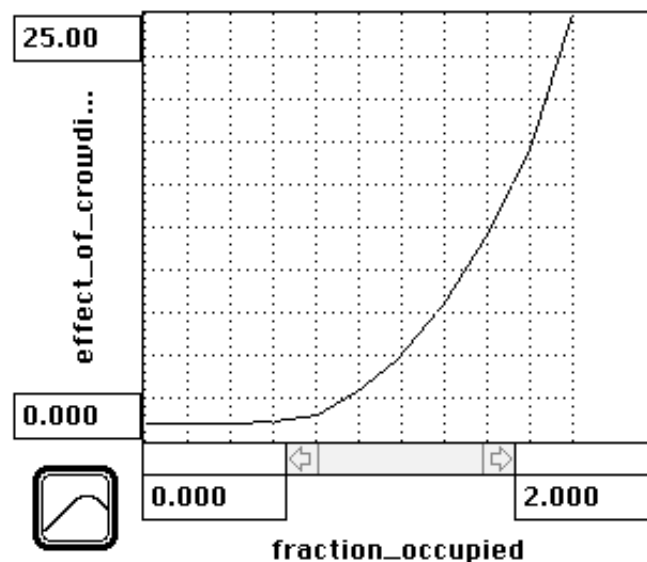

Figure 8: A robust, dimensionless table function

Now, if the aardvark population rises above the MAXIMUM AARDVARK POPULATION (if the "fraction occupied" is greater than 1.0) then the "effect of crowding" drastically increases the "death fraction" and the aardvark population stabilizes. The table function is ready to adjust the crowding multiplier accordingly for a whole range

of realistic parameter values. I ran the model again with a "BIRTH FRACTION" of 0.4 per year and obtained the graph in Figure 9.
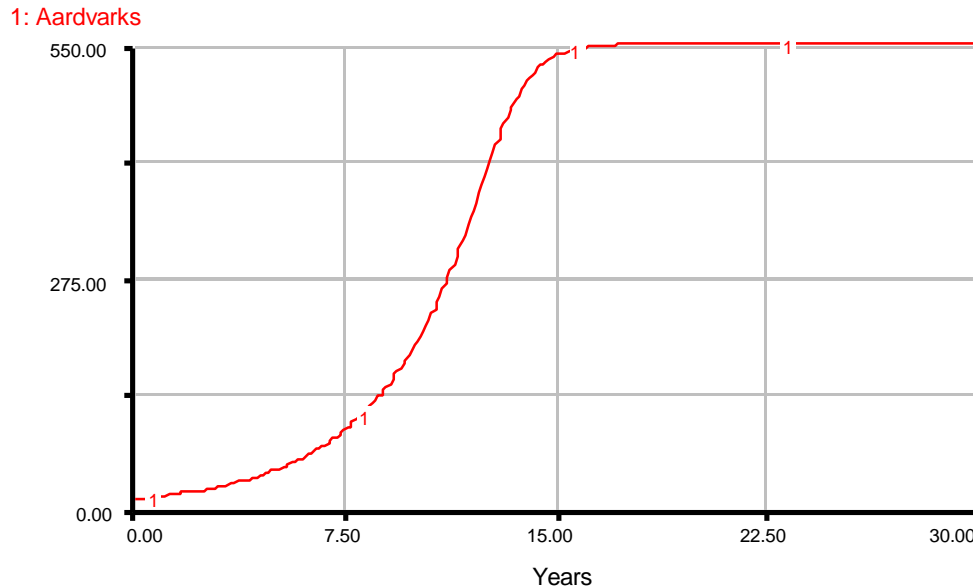

Figure 9: The corrected model run with a "BIRTH FRACTION" of 0.4 per year

I printed out another copy of my model with the new table function and a set of runs with different parameter values. All of the runs showed S-shaped growth. The next morning I handed my model to the professor for a third time. He checked the papers carefully, congratulated me for my perseverance, and then complimented me on a job—finally—well done.

7. KEY LESSONS

Table functions are useful for incorporating nonlinear relations into models. They allow modelers to easily specify nonlinear relationships between variables visually rather than algebraically. To be effective, however, table functions must be dimensionally consistent and robust. Table functions should have dimensionless inputs and outputs. Modelers nondimensionalize inputs with the use of normals. Normals are constants that act as a reference point for dynamic behavior. Multipliers, on the other hand, reflect the changing conditions within the system. To be robust, table functions must also extend to

cover the entire range of reasonable parameters values. The behavior of most real world systems is often insensitive to changes in parameters. For a model to be robust, the behavior that it generates must be as insensitive to changes in parameters as is the behavior of the system being modeled.

8. APPENDIX: EQUATIONS FOR THE CORRECTED MODEL

☐**Aardvarks(t)** = Aardvarks(t - dt) + (births - deaths) * dt
        INIT Aardvarks = 10
        DOCUMENT: the number of aardvarks in the current population
        Units: aardvarks

        ☞INFLOWS:
        Births = BIRTH_FRACTION*Aardvarks
        DOCUMENT: the number of baby aardvarks born each year
        Units: aardvarks/year

        ☞OUTFLOWS:
        Deaths = death_fraction*Aardvarks
        DOCUMENT: the number of aardvarks that die each year
        Units: aardvarks/year

○**BIRTH_FRACTION** = 0.3
        DOCUMENT: the fraction of aardvarks each year that give birth
        Units: aardvarks/aardvark/year or 1/year

○**death_fraction** = effect_of_crowding*NORMAL_DEATH_FRACTION
        DOCUMENT: the fraction of the population of aardvarks that dies each year.
        Units: aardvarks/aardvark/year or 1/year

○**fraction_occupied** = Aardvarks/MAXIMUM_AARDVARK_POPULATION
        DOCUMENT: the fraction of the population of aardvarks that dies each year.
        Units: aardvarks/aardvark/year or 1/year

○**MAXIMUM_AARDVARK_POPULATION** = 500
        DOCUMENT: the maximum number of aardvarks that can comfortably live in a
        1000 acre savannah
        Units: aardvarks

○ **NORMAL_DEATH_FRACTION** = 0.1
        DOCUMENT: the fraction of the population of aardvarks that die each year when
        natural resources are abundant

Units: 1/year

⊘ **effect_of_crowding** = GRAPH(fraction_occupied)
(0.00, 1.00), (0.2, 1.00), (0.4, 1.00), (0.6, 1.05), (0.8, 1.50), (1.00, 3.00), (1.20, 5.00), (1.40, 8.00), (1.60, 12.0), (1.80, 17.0), (2.00, 25.0)
DOCUMENT: When few aardvarks roam the savannah, crowding has little effect on the death fraction. As the fraction of land and resources occupied by the aardvarks increases, the effect of crowding increases, increasing the death fraction.
Units: dimensionless

Vensim Examples:

Mistakes and Misunderstandings: Table

Functions

By Aaron Diamond

January 2000

INITIAL AARDVARKS

Aardvarks

births

deaths

BIRTH FRACTION

fraction occupied

death fraction

effect of crowding

MAXIMUM AARDVARK POPULATION
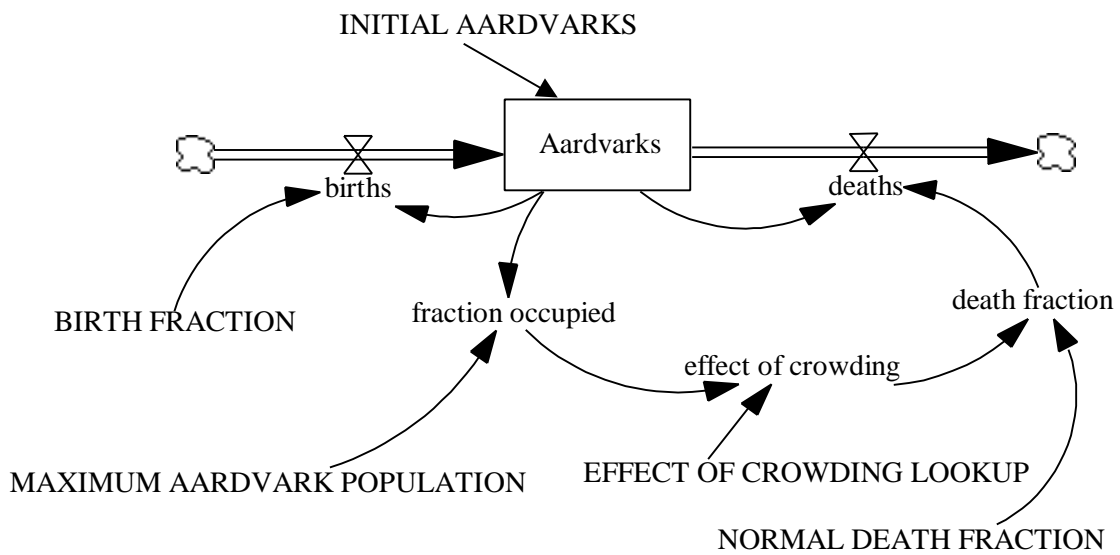
EFFECT OF CROWDING LOOKUP

NORMAL DEATH FRACTION

Figure 10: Vensim Equivalent of Figure 5: My second model with normals and a multiplier

Documentation for Aardvark model:

(01)   Aardvarks= INTEG (births-deaths, INITIAL AARDVARKS)

Units: aardvarks

The number of aardvarks in the current population.

(02)   BIRTH FRACTION=0.3

Units: 1/year

The fraction of aardvarks each year that give birth.

(03)   births=BIRTH FRACTION*Aardvarks

Units: aardvarks/year

The number of baby aardvarks born each year.

(04)   death fraction=effect of crowding*NORMAL DEATH FRACTION

Units: 1/year

The fraction of the population of aardvarks that dies each year.

(05)   deaths=death fraction*Aardvarks

Units: aardvarks/year

The number of aardvarks that die each year.

(06)   effect of crowding= EFFECT OF CROWDING LOOKUP (fraction occupied)

Units: dmnl

When a few aardvarks roam the savannah, crowding has little

effect on the death fraction. As the fraction of land and

resources occupied by the aardvarks increases, the effect of

crowding increases, increasing the death fraction.

(07)   EFFECT   OF   CROWDING   LOOKUP=([(0,0)(10,40)],(0,1),(0.2,1),   (0.4,1),
        (0.6,1.05),(0.8,1.5),(1,3),(1.2,5),(1.4,8),(1.6,12),(1.8,17),(2,25))
        Units: dmnl

(08)   FINAL TIME = 30
        Units: year
        The final time for the simulation.

(09)   fraction occupied=Aardvarks/MAXIMUM AARDVARK POPULATION
        Units: dmnl
        The number of Aardvarks relative to the maximum number of
        aardvarks that can comfortably live in the 1000 acre savannah.

(10)   INITIAL AARDVARKS=10
        Units: aardvarks

(11)   INITIAL TIME = 0
        Units: year
        The initial time for the simulation.

(12)   MAXIMUM AARDVARK POPULATION=500
        Units: aardvarks
        The maximum number of aardvarks that can comfortably live in a
        1000 acre savannah.

(13)   NORMAL DEATH FRACTION=0.1
        Units: 1/year
        The fraction of the population of aardvarks that die each year
        when natural resources are abundant.

(14)   SAVEPER = TIME STEP

Units: year

The frequency with which output is stored.

(15)   TIME STEP = 0.0625
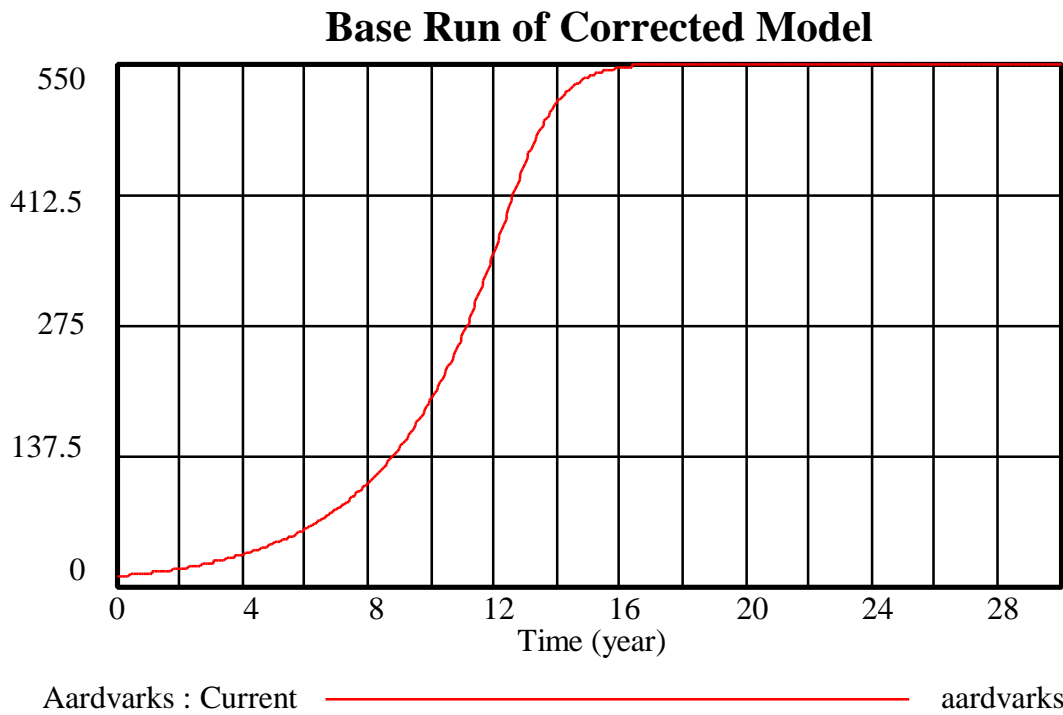
Units: year

The time step for the simulation.

Base Run of Corrected Model

Figure 11: Vensim Equivalent of Figure 9: The corrected model run with a "Birth Fraction" of .4 per year