The Bank Balance Problem

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Abstract

Why is the bank balance model important? Simple models discuss basic ideas of system dynamics, and a profound mastery of these concepts is indispensable to mastering more subtle and complex notions in any field. From the point of view of teaching mathematics, exponential growth could be taught using this model. This model can also serve as an introduction to the concept of limits and differential equations. From a systems point of view, many issues such as modeling, and positive feedback loops can all be dealt with just using this simple model.

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by
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Introduction

Seventy-two years ago, a wise man named Ralph put his entire fortune, two hundred dollars, in a secure bank. He never told anyone, not even his family, and all his descendants lived miserably all their lives. A few days ago, Joe, his grandson, found an old bank document stating when and where the money was deposited. Interestingly, the bank document affirmed that the money was to accrue at a constant interest rate of 6% during the entire duration of time that the money would be deposited. This bank was not affected by the great depression of the 1930's and the account remained open the entire 72 years. Joe, who's formal schooling was limited to elementary school, was unable to calculate how much money was presently in that account. Consequently, he decided to go and see his friend Alphonso, who was very good in math.

Causal Loop

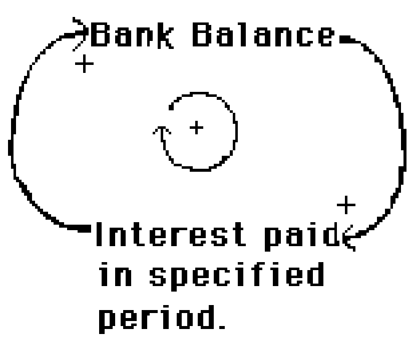

Figure 1. Causal Structure

Ralph told Alphonso about the bank account and how he was puzzled about how to determine how much money it contained. Alphonso's immediate reaction was to draw the diagram above. He called it a Causal Structure. The idea behind it was very simple. When

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there is an increase in the bank balance, there is an increase in the amount of interest paid per year. This is because, the interest paid per year is always going to be 6% of the bank balance. Because there is a linear[2] relationship between interest and the bank balance, an increase in the bank balance leads to an increase in the flow of interest. The interest is added to the bank balance. So when the interest paid increases, the bank balance increases even faster. This cycle continues causing the bank balance to get bigger and bigger. This behavior is called exponential growth. Exponential growth will be discussed later.

Because it was Sunday, the bank was closed, and they could not find out how often every year, interest was calculated. Alphonso, who had just bought an Apple Macintosh computer and the STELLA[3] computer package, decided to use them to help Joe visualize the process by which money in a bank account accumulates.

Illustration of the Model on STELLA

Transferring the causal loop connections into a STELLA model structure, Alphonso created the following model.

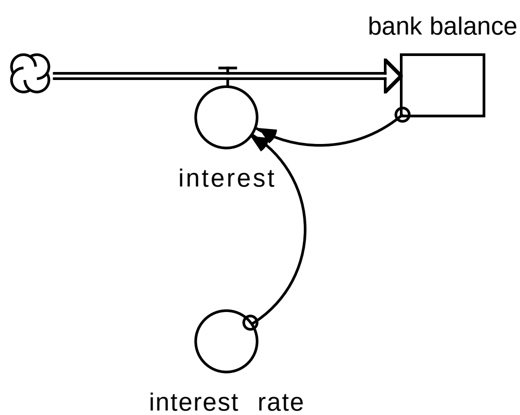


Figure 2. Stock and Flow Diagram of a Bank Account Accruing Interest.

The model has one stock where the money (bank balance) is accumulating. Alphonso reminded Ralph that a stock does not change instantly, it is either raised or lowered by a flow. The flow, represented by a pipe with a valve hanging down from it, is in this

[2] Linear meaning that in the relationship, an increase in one element is proportional to an increase in the other element. The increase in the bank balance is proportional to the increase in the interest paid.

[3] STELLA is a simulation Software package designed by High Performance Systems.

case the interest, and represents an inflow of money pouring into the bank balance. The inflow is regulated by the equation in the flow which states that interest is equal to the present bank balance times the interest rate.

interest = (interest_rate*bank_balance)

Converters are mainly used to hold constant values, to do algebraic operations, or to make conversions from one unit of measure to another. In this model, the converter is used to hold a constant value, the interest rate. The next question is how does the model work?

DT Solution Interval

DT (from Delta Time) is the interval of time between calculations in a model. It therefore answers the question of how many times in a time period the numerical values in a model are re-calculated. If your model has flows of cars, and the time unit is months, then the unit of measure for your flows will be cars/months. If DT is set to 1.0, a round of calculations will be done once every month. But if DT is changed to .5, the round of calculations will be carried out every 1/2 of a month. Similarly, if DT is changed to .25, the round of calculations will be carried out every 1/4 of a month.. In STELLA, DT is found in Simulation Time...under the specs menu. At every increment DT, STELLA carries out the necessary calculations. It first estimates the change in the stocks over the interval DT, and calculates the new values for these stocks using the previous values for the flows and converters. Then, using the new values for the stocks, it calculates new values for flows and converters. Finally it updates the simulation time by an increment of DT. In the bank balance problem, DT is initially equal to 1.0. The time unit is years.

INFLOW EQUATION
Interest Compounded Annually

The compounding frequency is how often per year interest is compounded. The next few pages show examples of differing compounding frequencies. DT under Simulation Time in the Specs menu should be set to 1.0. The only stock is the bank balance, and the only flow is interest[4]. STELLA first estimates the change in the

[4] It is important to differentiate between interest and interest rate. Interest is a flow that is equal to the interest rate multiplied by the bank balance. The interest rate is a constant equal to 6% annually.

bank balance stock over the time interval DT, by multiplying together the net flow and DT. The net flow in general is equal to the inputs to the stock minus the outputs from the stock. In this case, the net flow is equal to the interest rate multiplied by the bank balance. This net flow is the input to the bank balance stock. There is no output from the stock. Another example of net flow can be seen in a population model , where the net flow is births minus deaths. The stock integral equation for population is:

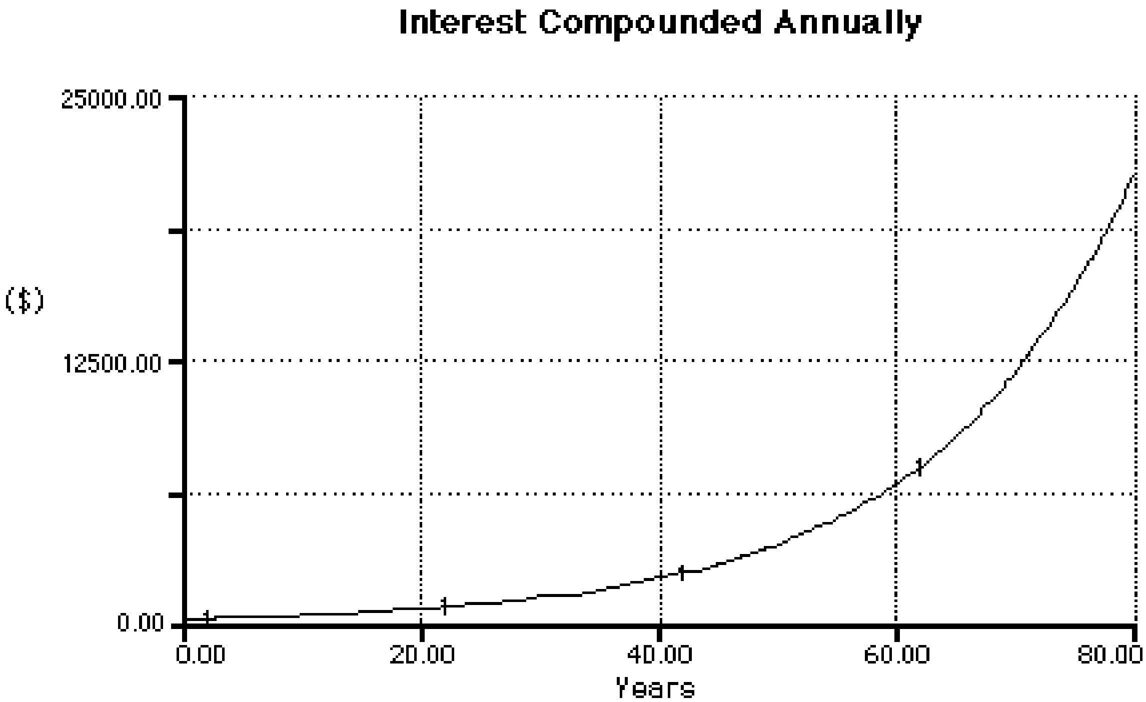
population(t) = population(t-DT) + DT*(births-deaths)

It is important to notice that the net flow is multiplied by DT here as well.

In the bank balance model, since DT is equal to 1.0, the change in the bank balance stock is the interest rate multiplied by the bank balance. STELLA then adds the bank balance and the change in the bank balance, and makes this the new value for the bank balance stock. Next, STELLA calculates the new value for interest by multiplying the updated bank balance and the interest rate. Finally it increments the simulation time by DT, and repeats the whole procedure. It keeps on repeating this procedure until the simulation time reaches the desired ending time. In real life, if interest is to be compounded annually, for example on December 31, it is calculated by multiplying the interest rate and the current bank balance. That interest is then immediately added on to the current bank balance, and this sum becomes the new bank balance. Every year, at the same time, the process is repeated. The figure below shows the progression of the bank account as interest compounded annually is reinvested into the bank account.
The equations governing the graph on the next page are:

interest = interest rate * bank balance
bank balance(t) = bank balance(t-dt) + dt*interest

where dt is 1.0 in the case of interest compounded annually.



(figure 3)

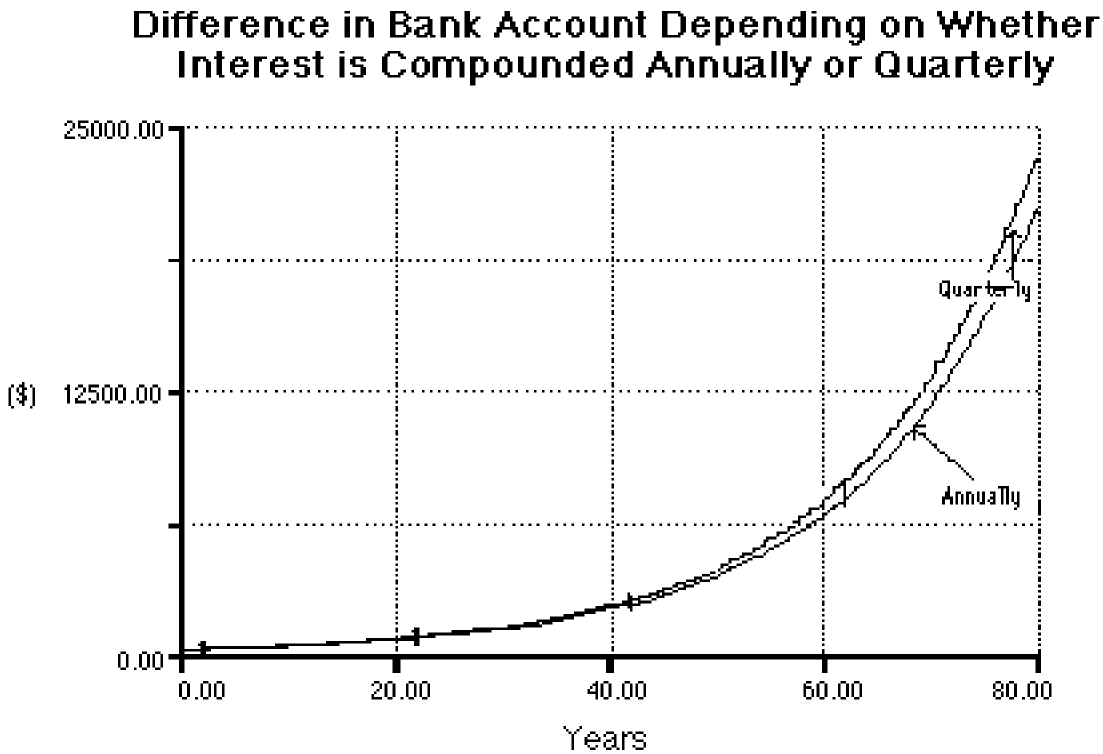
Interest Compounded Quarterly

When interest is compounded quarterly, the only difference is that the procedure discussed above will be executed four times every year instead of once. Accordingly, DT will have to be changed to .25. This would mean that when the simulation time would have reached 1 year, STELLA would have carried out the procedure four times, or once every quarter. The figure below shows the progression of the bank account as interest compounded quarterly is reinvested into the bank account.

The equations governing the graph below are:

interest = interest rate * bank balance
bank balance(t) = bank balance(t-dt) + dt*interest

where dt is .25 in the case of interest compounded quarterly.



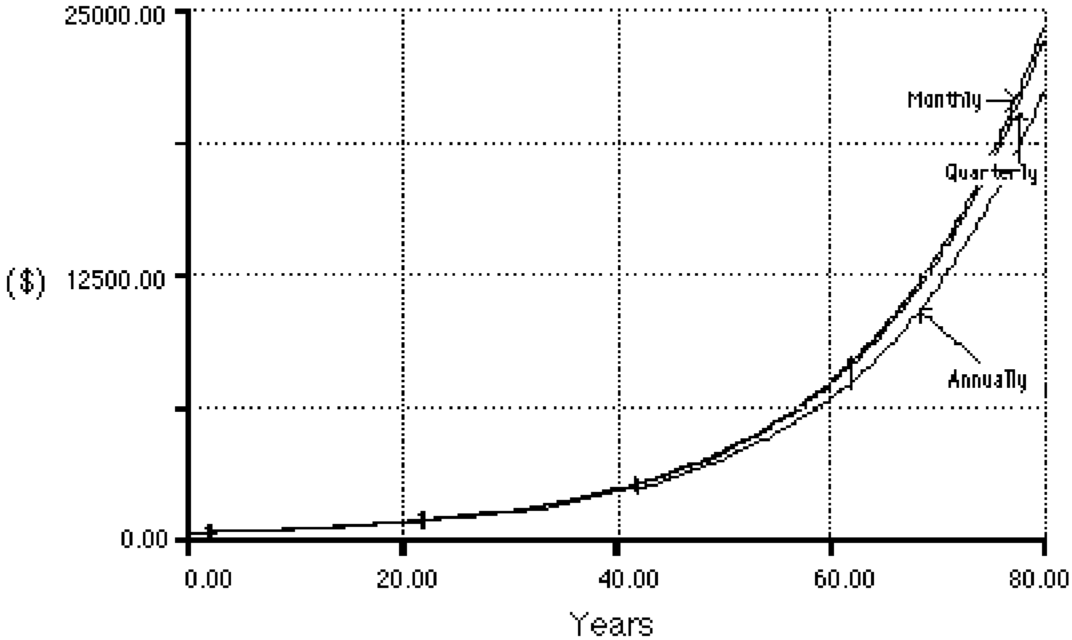
Interest Compounded Monthly

If interest is compounded monthly, once again the only change will be DT which will be set to 0.083333 (approximately 1/12). In this case, when the simulation time is equal to 1 year, the interest procedure will have been executed 12 times, or once every month. The figure below shows the progression of the bank account as interest compounded monthly is reinvested into the bank account. Again the equations governing the graph below are:

interest = interest rate * bank balance
bank balance(t) = bank balance(t-dt) + dt*interest

where dt is .083333 in the case of interest compounded monthly.

Difference in Bank Account Depending on Whether Interest is Compounded Annually, Quarterly, or Monthly.



Numerical Illustration of Ralph’s Problem

After explaining the procedure by which interest accumulates in a bank account, Alphonso then proposes to explicitly solve Ralph’s problem. Plugging in $200 for the initial bank balance, assuming the interest rate is 6% compounded annually, 72 years later, can you find out how much money the account would contain? What about 8 years later? Try determining how much money the account would contain after 72 years and after 80 years, for the three compounding frequency cases, annually, quarterly, and monthly.

Table for Answers

	Annually	**Quarterly**	**Monthly**
72 years later	________	_________	_________
80 years later	________	_________	_________

When interest is compounded annually, 72 years later the account would contain $13,279.52. Better yet 8 years later that amount will have reached a level of $21,159.20 . Ralph cannot believe his eyes! If interest is compounded quarterly, changing DT to .25 is the only necessary modification. When interest is compounded quarterly, Joe finds out that 72 years later he would have $14,562.98 in the bank. If he decides to leave the money in the bank just eight more years he would have $23,451.12. Joe now really hopes that interest is compounded quarterly.

But Alphonso, who is quite familiar with his math, tells him that he would get even more money if the interest is compounded monthly. In this case, DT will be set to .083333 (12 months in a year). Ralph would have $14,876.73 72 years after the bank account was first opened. He would have $24,013.16 8 years later. Joe leaves Alphonso happier than he had ever been, thinking about what he would buy tomorrow when he gets his money.

General Table

Compounding Interval	Annually	Quarterly	Monthly
Final Amount at 72 years	$13,279.52	$14,562.98	$14,876.73
Final Amount at 80 years	$21,159.20	$23,451.12	$24,013.16

(table 1)

Model Visualized as a Positive Feedback Loop

The bank balance model is a representation of a positive feedback loop. The bank balance continually feeds itself through the flow. After each time increment DT, the bank balance increases by the compounded interest. As in the case of interest in a bank account, the populations of rabbits, bacteria, and humans are examples of positive feedback loops because the bigger the population, the more births there are, the bigger the population becomes. The loops self-reinforce.

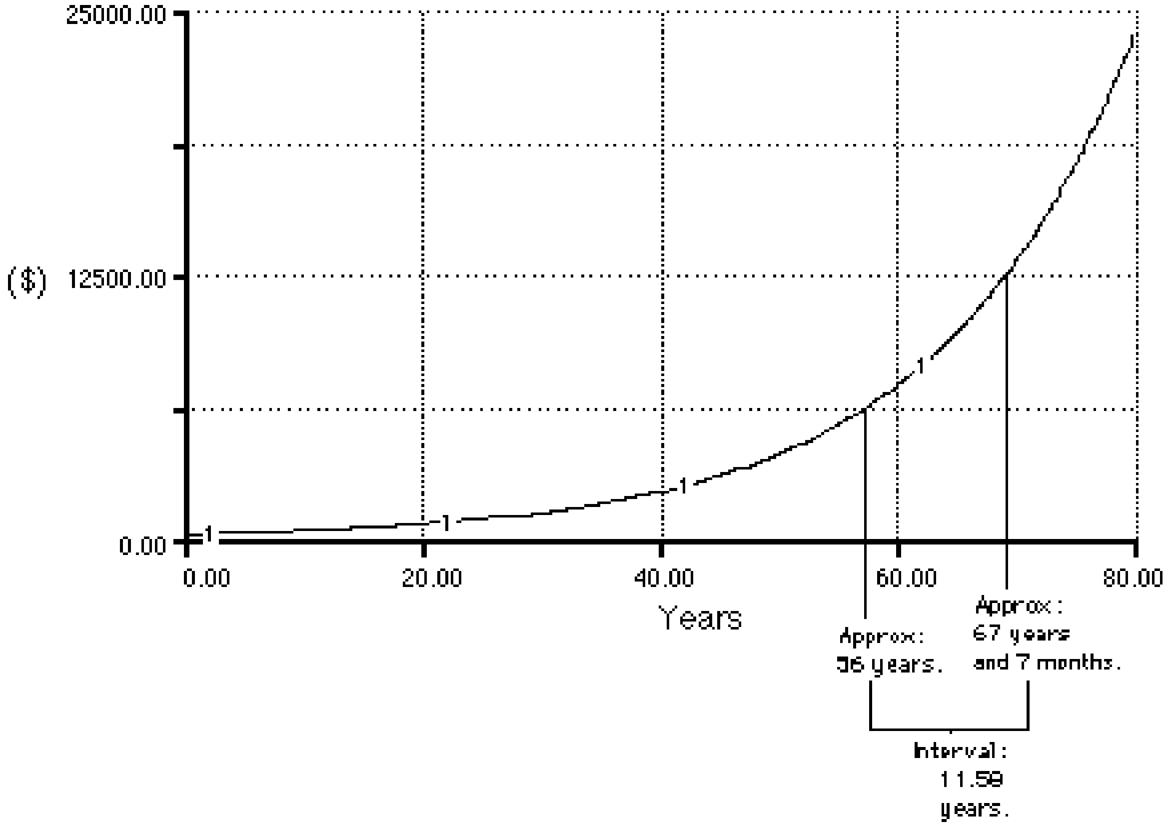
Introduction to Limits

As interest is calculated more frequently in a specified period, say one year, the total interest accrued over the year approaches a limiting value. This concept can be experimented with using a STELLA Computer package and a Macintosh. You could make DT smaller and see the effect on the bank balance after a certain period of time. Using this model, you could try changing DT to .00274 (1/365, interest compounded daily).

The idea of limits can also be visualized from the perspective of exponential growth. Mathematically, exponential growth is characterized by unbounded growth at a faster and faster rate. The quantity under study, in this case the quantity of money in the bank account, doubles repeatedly, after a specific time interval, called the "doubling time". As some stock in a positive feedback loop begins to increase, a snowball effect takes place and that quantity continues to increase at a faster and faster rate. In the case of the bank balance, when interest was compounded annually, the "doubling time" for the quantity of money in the bank account was approximately 11.95 years. In the case where interest was compounded quarterly, the "doubling time" was approximately 11.74 years, and in the case where interest was compounded monthly, the "doubling time" was approximately 11.58 years. As interest is compounded more often, the differences in doubling time get smaller and smaller, until the doubling time is very close to 11.55 years. Any amount of money that accrues at an interest rate of 6% annually, cannot be doubled in less than 11.55 years, independently of how often interest is compounded[5]. Other examples where exponential growth occurs is in the growth in population of bacteria or rabbits in an unlimited area, and in the growth of the world population in the last several centuries.

[5] See section of appendix dealing with limits for a better understanding of why this is so.

Illustration of the Notion of Doubling Time when Interest is Compounded Monthly



In approximately 11.58 years, the bank balance doubled from $6250 to $12500.

Appendix

Mathematical Approach to Limits

The idea of limits is a very important concept in math, and it shall be briefly introduced in this paragraph. Assume that interest is calculated X times in a year. Let p_k be the amount of money in the bank at an arbitrary time. Time X later, the amount of money in the bank will be p_{k+1}. In other words:

$$p_{k+1} = p_k + 0.06(1/X)p_k.$$

Moving p_k to the left and dividing by (1/X) we get:

$$(p_{k+1} - p_k)/(1/X) = 0.06p_k.$$

Let

$$dp = p_{k+1} - p_k$$

and

$$dt = 1/X.$$

The purpose of this exercise is to calculate interest an infinite amount of times in a given time, so as to determine the maximum (limit) amount of money that the bank account can contain after a specified period of time. As interest is calculated more and more often in one year , X gets bigger and bigger, until it approaches infinity. When this happens, dt approaches 0, because 1 divided by infinity is so small that it is very close to 0. Expressed as a differential equation, the equation is:

$$dp/dt = 0.06p.$$

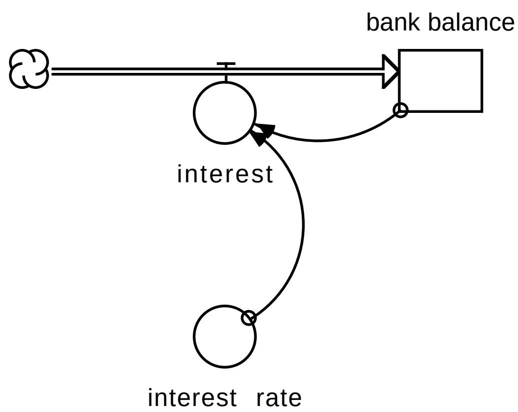
The solution to this differential equation is:

$$p(t) = e^{0.06t}p_0.$$

In Joe's case, as discussed earlier, over a period of 72 years, even if interest were calculated infinitely many times, presently, he would have $15,037.73, not a great difference from when interest was compounded monthly.

Equations and Documentation

Model



bank_balance(t) = bank_balance(t - dt) + (interest) * dt
INIT bank_balance = 200

DOCUMENT: The value was chosen arbitrarily. 200 dollars was chosen because it is a simple number to work with.

interest = interest_rate*bank_balance
DOCUMENT: This is the inflow into the bank balance. The interest is compounded every time interval DT and then added to the bank balance. Its units are dollars/time.

interest_rate = .06
DOCUMENT: 0.06 is actually a 6% per year rate. This rate can also be arbitrarily modified.