Designing a Hands-On Approach to Calculus

by

William B. Glass

Submitted to the Department of Electrical Engineering and Computer Science
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In this thesis, I describe a computer-oriented "hands-on" approach to teaching calculus. This is based in part upon a course I taught to eight high school students in the Spring of 1992 entitled "Calculus Can Be Fun! A Hands-On Introduction to Integration." This course used physical and computer experimentation, group work, discussion, and writing as tools for instruction. This thesis both describes the overall design of the course and focuses on one unit teaching The Fundamental Theorem of Calculus.

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Chapter 1

Calculus: Purposeful or Problematic?

1.1 What is Calculus?

Simply put, calculus is the mathematics of change. Like all branches of mathematics, calculus is also a language of description, modeling, and problem-solving. As Gilbert Strang firmly states in the first paragraph to his text *Calculus*:

> Calculus is a course with a purpose. It describes growth and decay and change. Population has a growth rate – which changes. The value of money has a decay rate – which may grow. Whatever happens in life, the one sure thing is change. To understand and model those changes, that is where calculus is needed and used. [1]

Calculus occupies an important role in twentieth century education. It is the capstone in a mathematical learning sequence that follows arithmetic and algebra. For mathematicians, it opens a door into a higher area of mathematics concerned with analysis and abstract operations. For scientists and engineers, it provides a powerful set of tools to explore their discipline. For many Americans, however, it probably is the last mathematics course they will take.

Historically, calculus has a rich history of mutually enriching growth with the sciences, particularly physics. The subject began when Newton needed a way of math-

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[1]Strang, G. (1991). *Calculus*, p. v. Wellesley-Cambridge Press.

ematically describing his laws of motion in the seventeenth century. Today, it continues to grow with the advent of computers and new research in dynamic phenomena. Except for recent years, however, the content of college calculus courses has changed little over the last century. [2]

1.2   Problems of Contemporary Calculus Pedagogy

1.2.1   Calculus by Rote

The universal nature of calculus helps and hinders the design of an effective calculus course. Calculus describes processes of change everywhere around us. However, this generality makes it necessary to use abstruse symbols and notation in order to capture all possible situations. This added complexity contributes to making calculus a difficult and time-consuming course.

One way that traditional calculus courses and textbooks have chosen to simplify the teaching of calculus is to emphasize the solving of clearly-defined problems with methods involving pattern matching and the mechanistic application of formulas. For example, one major textbook says this when discussing integration:

> A major step in solving any integration problem is recognizing which basic integration formula to use to solve the problem. Skill in recognizing which formula to use requires memorization of the basic formulas and lots of practice in using them. [3]

This practice may lead to success by the student at solving given problems on problem sets or exams, but does not necessarily lead to proficiency at modeling vague problems from other disciplines or problems that the student may encounter in the real world.

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[2]Strang, G. (1987). Calculus for a Purpose. *Calculus for a New Century: A Pump, Not A Filter, 54.* Mathematical Association of America. Washington, DC.

[3]Larson, R. E., & Hostetler, R. P. (1986). *Calculus with Analytic Geometry, 434.* D.C. Heath & Co. Lexington, MA.

1.2.2   Mode of Instruction

The current mode of instruction of calculus also can cause many difficulties. A typical calculus course is organized around teacher-centered lectures, homework to be worked on independently by the student, and proctored exams. Students are usually either implicitly or explicitly discouraged from working together. There is almost always a unidirectional flow of information from the teacher to the student. In general, calculus courses are taught in only one manner to students with many different learning styles. Finally calculus is usually taught as in isolation. The problems that students usually solve are abstract. Few connections are drawn from these problems to ideas or exercises the student may have encountered in other disciplines.

1.2.3   Problems with Content

There are many potential problems with the content of the calculus curriculum. One area of concern is that calculus relies heavily on a number of topics of mathematics that students often understand poorly. For example, calculus textbooks often explain concepts with graphs, yet many students have trouble understanding the relationship between graphs and equations. [4] Another topic that students may have difficulty with is that of functions, which has been called "one of the most difficult concepts to master in the learning sequence of school mathematics." [5] Most previous courses have examined specific points in a function, such as the zeroes of a function. In calculus, students begin to work with the entire function at once. This can be confusing.

Topics do not even have to be complicated in order to cause problems. The idea of a variable, something that seems basic to most mathematicians, has aspects can perplex many students. [6]

The ordering of calculus topics can also often cause many problems. Typically, the ordering is on a mathematical "deductive" basis. Almost all textbooks teach limits followed by derivatives, followed by integrals. This follows from the manner in which

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[4]Leinhardt, G., Zaslavsky, O. & Stein, M. K. (Spring 1990). Functions, Graphs and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, Vol. 60, No. 1, 1-64*.

[5]Eisenberg, T. (1991). Functions and Associated Learning Difficulties. *Advanced Mathematical Thinking, 140*. Kluwer Academic Publishers. Norwell, MA.

[6]Epp, S. (1986). The Logic of Teaching Calculus. *Towards a Lean and Lively Calculus, 45*. Mathematical Association of America. Washington, DC.

each is taught. Integrals are defined in terms of derivatives, derivatives are defined in terms of limits.

Research shows that students have many difficulties learning limits as the first step in calculus. [7] They often have little experience dealing with necessary pre-requisites such as continuous functions or formal mathematical definitions. The notion of limits presents severe challenges to the intuition of the student (try answering the question in fifty words or less: is a limit ever attained or not?) The informal explanations of limits are often vague and can give rise to many different interpretations.

Cornu says:

> It is important to consider the order in which the limit concepts are presented. Not only is there the question of designing a logical mathematical order of concepts, but also the cognitive appropriateness of the curriculum sequence and of the problems to be solved. It is now well-established that in the transition to advanced mathematical thinking a purely logical sequence of topics, in which the mathematical concepts are introduced through definitions and logical deductions, is likely to be insufficient. [8]

A different model of teaching limits might take in to account these varying perspectives and let students learn limits in context – learning limits as they needed them in differentiation or integration. Other possibilities include computer labs designed to let students explore the limit concept and special discussions to help students be aware of possible misconceptions and misunderstandings. Building on "basic principles" through abstract definitions may not be the best way for students to learn.

1.2.4 Understanding the Relationship between Rate and Amount

Earlier, I referred to calculus as "the mathematics of change." Yet both research and anecdotal evidence show that students often do not internalize the relationship between rates of changes and the amount that is changing. For example, the author has given many MIT students the following exercise that should be fairly simple,

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[7] Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, Vol. 22, No. 3, 219-236*.

[8] Cornu, B. (1991) Limits. *Advanced Mathematical Thinking*. Kluwer Academic Publishers. Norwell, MA.

especially after taking a calculus course.

   Given the graph of the velocity of a car through time (shown in in Fig-
   ure 1.1), draw a rough corresponding graph of the position of the car through
   time.
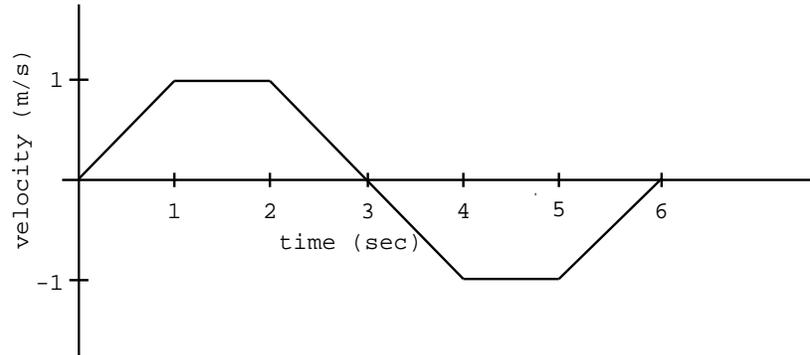
Figure 1-1: *If this is the velocity of a car, what is the position of the car over time?*

   Most of the respondents answered with a variant of the graph shown in either Fig-
ure 1.2 or Figure 1.3. Very few responded with the correct graph shown in Figure 1.4.
Notice that the key mistake in Figure 1.2 is that when the speed of the car begins
to go down, the distance the car has traveled is shown as *decreasing*, while it should
be still still increasing. On the other hand, the author of Figure 1.3 does realize the
difference between a change in speed and a change in position, but still makes a major
error concerning the relationship between the slope of a graph and the rate of change,
one of the key concepts of calculus.

   Admittedly, this lack of understanding of the relationship between a rate of change
of an amount and the amount itself is part of a larger problem involving more than
calculus. It is usual, for instance, for newspapers to rail against the large deficit
caused by the federal government. However, they are missing the point, for the deficit
is only the yearly overexpenditure. The real monster is the accumulation of deficit–
the federal debt. Even if the government reduces the deficit to zero (by balancing
the budget) the federal debt has not been reduced at all (and consequently, is still
accumulating interest).
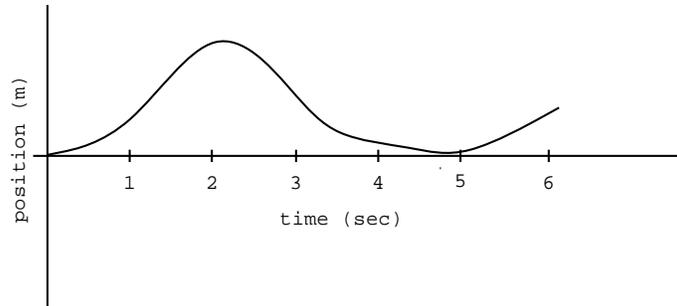
Figure 1-2: *One very incorrect answer*
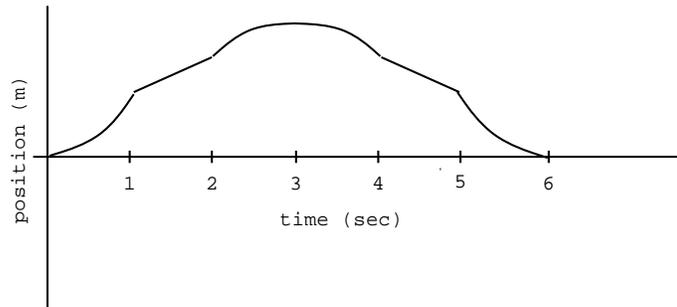
Figure 1-3: *Another incorrect answer*

1.3 Why the Call for Change?

1.3.1 A Pump, Not a Filter

In the last five years there has been an explosion of conferences, books and papers calling for a change in the way calculus courses are designed and taught. [9]

Robert M. White, attendee at a 1987 national colloquium entitled "Calculus for a

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[9]For a short timetable of some of the last decade's conferences and relevant publications, see "Evolution in the Teaching of Calculus" by Bernard R. Hodgson in *Calculus For A New Century: A Pump, Not A Filter, 49*
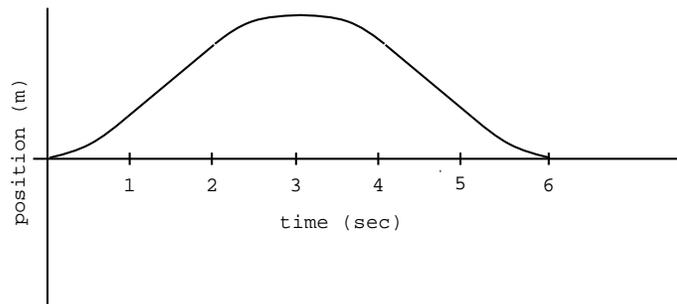
Figure 1-4: *The correct answer*

New Century" summed up the urgency in the quest for a better calculus in a presentation published in the conference proceedings:

> The national spotlight is turning on mathematics as we appreciate its central role in the economic growth of this country. The linkage between mathematics and economic growth needs to be made, and needs to be made stronger than it has been to date. Calculus is a critical way-station for the technical manpower that this country needs. It must become a pump instead of a filter in the pipeline. [10]

There are a number of reasons for the growing realization that a large change is needed. Many of the these are fairly recent.

1.3.2 Advances in Computer Technology

In the last few years, electronic computers and graphing calculators have changed both the breadth of mathematics available to students as well as the manner in which the mathematics can be experienced.

Computers are fast and interactive. They can act as time compressors, performing at a high speed numerical and symbolic computations that would take a long amount of time to perform by hand. With computers, mathematics becomes more interactive (and often experimental). Students can play "what if?", performing repeated calculations with different parameters. The ability of computers and calculators to quickly graph results of computations can also help students to visualize solutions to problems.

The current availability of relatively inexpensive computers and graphing calculators has widespread implications for content, method of instruction, and assessment. It is difficult to use computers without radical changes to each of these areas of calculus instruction. Computers do not only make it possible for change in calculus to occur, they make it necessary. [11]

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[10]White, R. M. (1987). Calculus of Reality. *Calculus For A New Century: A Pump, Not A Filter, 9.* Mathematical Association of America. Washington, DC

[11]Renz, P. (1986). Steps Toward a Rethinking of the Foundation and Purposes of Introductory Calculus. *Towards a Lean and Lively Calculus, 104.* Mathematical Association of America. Washington DC.

1.3.3 New Research in Mathematics of Dynamic Systems

As computers have become more prominent the field of mathematics has seen many new developments. The ability of computers to perform iterative calculations has led to many advances, particularly in areas concerning dynamic systems. Some of these areas are fluid dynamics, quantum field theory, mathematical biology, and principles of wave scattering and diffraction. The publication of images of fractals and the use of terms such as "chaos" has stirred the imagination of the public. Introductory calculus courses can capture some of this excitement by taking advantage of computer modeling and simulation to explore these areas. [12]

1.3.4 New Trends in Mathematics Pedagogy

In the past decade many mathematical educators have proposed shifts in the manners teachers organize their classrooms and relate to their students. Principal shifts are towards collaborative, small-group learning; using manipulatives and other hands-on materials; and teaching mathematics in context, using specific problems, with connections to general principles of mathematical reasoning.

Some of these new directions are perhaps best embodied in the National Council of Teachers of Mathematics (NCTM) Professional Standards for Teaching Mathematics. The NCTM advocates working:

- toward classrooms as mathematical communities– away from classrooms as simply a collection of individuals;

- toward logic and mathematical evidence as verification– away from the teacher as the sole authority for the right answers;

- toward mathematical reasoning– away from merely memorizing procedures;

- toward conjecturing, inventing, and problem-solving– away from an emphasis on mechanistic answer-finding;

- toward connecting mathematics, its ideas, and its applications– away

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[12]Lax, P. (1986). In Praise of Calculus. *Towards a Lean and Lively Calculus, 1.* Mathematical Association of America. Washington DC.

from treating mathematics as a body of isolated concepts and proce-
dures. [13]

1.4 An Experiment in Calculus Instruction

In reaction to both the problems of traditional calculus courses and the call for change,
I investigated an alternate model of calculus instruction in the Spring of 1992. I de-
signed and taught a "hands-on" course to eight high school students entitled "Calculus
Can be Fun! A Hands-on Introduction to Integration. " Following a student-centered,
experimental perspective, the course utilized physics and computer-oriented exper-
iments, group problem-solving sessions, class discussions, and writing exercises to
study principles of integration. While the course introduced concepts of calculus, it
was not intended to directly replace existing high school or college classes.

This course included many ideas studied in current research on calculus instruc-
tion. Chapter two of this document discusses some of these ideas related to computers
and calculus. Chapter three presents information on the content and design of "Calcu-
lus Can Be Fun!" Finally, chapter four describes one lesson in detail. Handouts given
in the class can be found in the appendix.

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[13]National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathe-
matics, 3.* Reston, VA.

Chapter 2

Computers and Calculus

2.1   Introduction

Up until the last decade, hardware and software limitations constrained the use of computer technology in higher-level math mainly to large-scale numerical computation. Recently, however, there have been many changes. Display technologies and programming has improved, allowing sophisticated two and three dimensional graphing. Computer algebra programs such as Maple and Mathematica are in common use. Personal computers are now widespread and easy to use. Sophisticated calculators have entered the market in the last few years that combine scientific functions with graphing and programming ability. [1]

Computers have many uses. They were originally designed as "number-crunching" machines; a prime use of early computers was calculating missile trajectories. Today, computers perform calculations in business finance programs, computer simulations, and scientific analyses. Computers also graph functions and can draw pictures. They can also be set up to allow students to program them with mathematical algorithms and ideas.

The speed and power of computer calculations have many implications for calculus problem solving. Unlike calculus problems designed for computation by hand (where the answer must be easy to calculate), the computer can perform computations of

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[1]As computers grow smaller and calculators grow bigger, the distinction between them blurs. In the rest of this paper (unless specified otherwise) when I use the term "computer" I am referring to both computers and calculators.

almost any complexity. For example, a student can enter a formula for numeric integration and a formula representing a curve to integrate. The computer will then rapidly perform the calculations and present the result to the student. The student gets immediate feedback as well as an opportunity to try again with a similar problem and observe the difference in results. With computers at hand, students can shift from applying mathematical algorithms to reasoning about mathematics.

Computers are tools; the educational benefit gained from their use depends on what role they play in the classroom. In particular, there are four major areas where computers can have an impact on students. Computers can help students to:

1. Understand the relationship between functions and their graphs

2. Use mathematics at different levels of abstraction through computer algebra systems.

3. Enable students to "program" calculus rules and algorithms.

4. Model and simulate dynamic systems

2.2   Functions and Graphs

Understanding the relationship between functions and graphs is an important element of both calculus and pre-college mathematics. However, students have many difficulties working with a number of different aspects of functions and graphs. Computer graphing programs are a tool that can be used to help resolve some of these problems.

Many mathematics programs have the ability to graph functions and sets of data. These programs have many features and uses. With them, students have the power to choose arbitrary (and realistic) numbers. Students can "play with functions" – it is easy to graph and compare similar functions. Students can quickly graph the results of computations, simulations, or experiments. They can use graphical methods to qualitatively solve mathematical problems.

To create a graph of a function, students do not have to consider every detail of creating the graph. Graphs contain much information both of a quantitative nature ("the line has a y-intercept of 4.") and a qualitative nature ("polynomial functions grow slower than exponential functions."). Research suggests that most teaching of

graphs is confined to studying quantitative aspects. However, computers can free students from the technical and computational aspects of graphing, allowing them to concentrate on interpreting higher-level "qualitative and global" features. [2]

The opportunity to learn and use qualitative graphing skills has ramifications throughout all of calculus. Demana points out that most calculus textbooks assume students have "control" of graphing. Calculus textbooks explain many aspects of calculus with graphs and diagrams. These explanations may be incomprehensible to many students due to their lack of familiarity with graphing functions. The use of function graphing programs may be one way to help students become more comfortable with functions and their graphs as well as help students visualize key calculus concepts. [3]

2.3 Abstraction and Mathematics

Computer Algebra Systems (CAS) are also useful in the calculus classroom. Examples of current CAS's include SMP, Mathematical, MACSYMA, and Maple. CAS's allow students to transform and evaluate mathematical expressions. More specifically, a CAS is a program that will solve symbolically solve a calculus problem, for example, solving the indefinite integral:

$$p(t) = \int t \, \sin t \, dt$$

Most CAS's also include sets of graphical and numerical tools, providing powerful but flexible means of representing mathematical concepts.

CAS's can usually be operated in two manners. They can be used in "calculator mode" with a series of one-line input commands, or they can be programmed to perform sophisticated mathematical computations. (The former is the usual mode of operation. The latter has important ramifications discussed later in this chapter) While most CAS's are powerful enough to manipulate complicated mathematical expressions, beginning calculus students can easily learn to use them at an appropriate

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[2]Leinhardt, G., Zaslavsky, O. & Stein, M. K. (Spring 1990). Functions, Graphs and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, Vol. 60, No. 1, 12.*

[3]Demana, F. (1988). Using Computer Graphing to Enhance the Teaching and Learning of Calculus and Precalculus Mathematics. *Proceedings of the Conference on Technology in Collegiate Mathematics: The Twilight of the Pencil and Paper.* Addison-Wesley Publishing Co. Reading, MA.

level. [4]

Computer Algebra Systems allows students to focus on the details of problem solving rather than on the details of computation. In other words, students can manipulate equations without manually going through all the necessary computations of each step. This "black box" approach can be rather controversial. An argument can be made that this method might lead students to use computers "mechanically" without understanding underlying concepts. However, a different perspective is that by selectively ignoring the specifics of lower-level manipulative skills, students can concentrate on higher-level reasoning. [5]

Using CAS's for "stepwise refinement" partially solves the above conflict. Problems can be examined at a number of different levels of abstraction, ignoring the details below that level. For example, a stepwise refinement can be used to help understand Simpson's Rule (a method of numerical integration involving approximating area under a curve with trapezoids).

To apply stepwise refinement to Simpson's Rule, students would use the CAS in a succession of different manners:

- Using a built-in Simpson's Rule operation to integrate the function.

- Writing a program to calculate the area of a trapezoid, dividing the area under the curve into trapezoids, and repeatedly applying the function.

- Calculating the summation the above program represents, and applying it to the function.

By going through all the steps with a Computer Algebra System, the student gets a "big picture" view of what Simpson's Rule is. By successively implementing at lower levels, the student comes to understand how it works. This is in contrast to most textbooks, which will explain each level, but only make students memorize the summation at the end of the derivation. [6]

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[4]Zorn, P. (1987). Computing in Undergraduate Mathematics. *Calculus For A New Century: A Pump, Not A Filter*. Mathematical Association of America. Washington, DC.

[5]Kenelly, J. (1987). Computer Algebra Systems. *Calculus for a new century: A pump, not a filter, 78*. Mathematical Association of America. Washington, DC.

[6]Devitt, J. S. (1988). Adapting the Maple Computer Algebra System to the Mathematics Curriculum. *Proceedings of the Conference on Technology in Collegiate Mathematics: The Twilight of the Pencil and Paper*. Addison-Wesley Publishing Co. Reading, MA.

2.4   Learning Calculus through Programming

Programming is the art of instructing a computer in how to perform a task. Programming can be done in a conventional computer language such as C or Fortran or by specifying a set of rules in a spreadsheet or modeling program.

A very different way of learning calculus involves the use of programming to learn fundamental concepts of mathematics. As students construct a program to solve a problem, they simultaneously construct their understanding of the mathematical principles involved.

A computer language that was designed to support such mental construction is ISETL. In ISETL, students can write programs that are almost identical to standard mathematical notation. For example, at Purdue University, students write programs to (numerically) compute a definite integral. Schwingendorf and Dubinsky describe this process.

> The key computer activity for the students here is to construct a number of *ISETL* funcs which implement various Riemann sums. Here is a `func` `RiemLeft` which implements a Riemann sum by using left endpoints of the subintervals.

```
RiemLeft := func(f,a,b,n);
    x := [a+((b-a)/n)*(i-1)
        : i in [1..n+1]];
    return %+[f(x(i))*(x(i+1)-x(i))
        : i in [i..n]];
    end;
```

> Notice that, except for using `%+` in place of $\sum$, the syntax here is correct mathematical notation. . . Of course there are variations corresponding to taking right endpoints, midpoints, minima, maxima. . . . The students apply these `funcs` to many functions, draw pictures on graphs produced by Maple and see that the mathematical calculation does approximate the area under a curve, or between two curves, for example. The standard properties of the integral (linearity, monotonicity, concatenation of intervals) are more

easily understood by the student because they correspond to properties of computer constructs which he or she has made. Even limits can begin to make sense for the student by writing appropriate programs. [7]

2.5 Modeling and Simulating Dynamic Systems

Modeling is also a means for students to make explicit their assumptions. Students (and adults!) share many misconceptions as to the nature of dynamic systems. Through modeling and simulation, these misconceptions can be reconciled with reality. Students use the computer to make explicit their assumptions about the relationships within a system. Computer simulation provides the means to calculate the dynamic consequences of those assumptions, without the need for complete understanding of higher-level mathematics. Similar to the technique of programming to learn calculus, building a model also builds an understanding in the mind of the student. [8]

Models can be entered in many forms. Many programs will calculate differential equations. Some programs, such as Extend or Dynamo, have their own simulation language. Others, such as STELLA, let the user represent dynamic systems graphically. Some programs have elaborate user interfaces that add realism and interest. Many programs have built-in models that are hidden "black boxes". These programs only allow the student to perform simulation.

One of the larger debates in calculus occurs between proponents of theory-oriented courses and of application-oriented courses. Modeling and simulation is a method to bring these two seeming opposites together. Students can learn both about a specific dynamic system as well as the characteristics it shares with other systems.

For example, students may model and simulate a cup of hot coffee cooling on a table. This is a specific example of a first order thermodynamic system. However, the exact same relationships exist within a can of water that has a hole in the bottom. The same goes for the discharge of a capacitor that is in parallel with a resistor. Each of these three systems is an example of first order *negative feedback*, with the dynamic

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[7]Schwingendorf. (1990). Calculus at Purdue. *Priming the Calculus Pump: Innovations and Resources.* Prepared by the CUPM Subcommittee on *Calculus Reform and the First Two Years*. Mathematical Association of America. Washington, DC.

[8]Forrester, J.W., (1991) System Dynamics– Adding Structure and Relevance to Pre-College Education, System Dynamics Group memo, D-4227-1.

behavior of exponential approach to an equilibrium state. (The coffee cup cools to the temperature of the room, the can of water pours out until it is empty, and the capacitor discharges until the voltage is zero [9]).

2.6   Revealing the Laboratory Nature of Calculus

The uses of computers in calculus that are listed above have strong implications for how calculus is taught. The most important ramification is that to use these tools effectively, calculus becomes a laboratory subject.

Computers help the student give meaning to concepts of calculus. They can help students construct an understanding by allowing the student to portray information in different forms (for example as a diagram, an equation, a table of numbers, or a graph). They provide an electronic workbench for the student to explore fundamental ideas – ideas that come initially both from others and the student. The computer allows students to set problems to solve, hold experiments, analyze results, and make conclusions– in short, to apply the scientific method to mathematics.

A final thought: given the radical transformations to a traditional calculus classroom that computer use involves, it is unsurprising that computers are used so rarely as they are.

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[9]This isn't quite true. Actually, the can of water empties until the water level is equal to the water level outside the can, as any child playing with a sieve in the bathtub knows. Similarly, the equilibrium condition of the capacitor also depends on the circuit the capacitor is embedded in

Chapter 3

Teaching Calculus

3.1 Introduction

During the Spring of 1992 I designed and taught a calculus class entitled *"Calculus Can Be Fun! A Hands-On Introduction to Integration."* This chapter gives an overview of that course. The next chapter presents a detailed description of the second unit in the course covering the Fundamental Theorem of Calculus.

The calculus class that I taught was organized under the auspices of the MIT High School Studies Program (HSSP) [1] and ran for ten Saturdays from March 7, 1992 to May 10, 1992. Each class session was about two hours in length. It was described in the HSSP catalog to students by the course description shown in Figure 3-1.

In this document I present the second lesson of my course. It does not match exactly what I taught, but the differences are mostly minor changes in method or style. These modifications are due to the fact that while HSSP presented a good opportunity to test the ideas contained here, it had constraints of time and attendance that limited the amount I could teach. In addition, as I taught the course I discovered areas of my curriculum that needed revision. Significant differences between the original course and the one presented here are noted when appropriate.

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[1] The High School Studies Program is one of several projects run by the MIT Educational Studies Program, a student activity at MIT. HSSP and its sister projects (Splash, Junction, and the SAT Preparation Program) give MIT students the chance to teach classes to Boston area high school students who come to MIT on weekends or the summer.

Calculus Can Be Fun
A Hands-On Introduction to Integration

Math does not have to be dry, boring or seemingly useless. Math doesn't necessarily involve lots of incomprehensible symbols, either. Instead, the processes and concepts that make up mathematics are the glue that hold our universe together. Almost everything we do involves math, even something as simple as operating a tap to fill a glass of water!

This course explores the topic of integration, traditionally taught in the second half of an introductory calculus class. Through problem sets, discussion, lab work, and computer experiments, we will investigate the principles behind integration, and how they describe fundamental processes behind many everyday experiences.

Note: This class *does not* follow the AP Calculus syllabus.

Prerequisites: A background in algebra is needed. Experience with trigonometry is preferred, but not required. Students with an extensive background (one year or more) in calculus will not be allowed to enroll. No computer experience is necessary.

Enrollment limited to 10 students.

Figure 3-1: HSSP course description

3.2   Goals for Instruction

As I designed my course, I considered many issues of teaching, learning, and calculus. I found that their were four basic pedagogical themes I wanted to emphasize. These were:

1. Doing versus listening

2. Cooperative learning

3. Learning in context

4. Mathematical reasoning

5. Appropriate use of technology

These issues are interrelated in a complex and subtle fashion that makes it difficult to separate them and discuss them in a linear fashion. However, I'll describe my view of each of these below.

3.2.1 Doing versus Listening

An oft-repeated quote is "Tell me and I will forget; Show me and I might remember; Involve me, and I will understand." This principle is key to the success of this approach to calculus.

Lynn Steen discusses the relationship between "doing math" (as opposed to "listening to math") and mathematical maturity in a paper entitled "Developing Mathematical Maturity." She concluded that students begin to become "real" mathematicians through doing math in context, not through only listening to mathematicians (which she refers to as traditional learning).

Doing math is easy in discrete math, she says, where students can program, study algorithms, and solve concrete problems. However, doing is more is difficult in the areas of continuous math.

> This is why calculus courses have evolved to be so much learning and so little doing. To be sure, all calculus students do lots of homework but they rarely do mathematics. Students in computer science, on the other hand, are often actively doing mathematics in the process of designing and implementing complex algorithms. . . The beginning courses in computer science can offer students a glimpse of doing new things, whereas the beginning exercises in mathematics only offer an endless sequence of exercises, mostly two or three centuries old. [2]

In contrast, much of "Calculus Can Be Fun!" is taken up by hands-on activities. Some involve physical experiments. Most involve the computer. By designing solutions to solve problems, students build an understand of the concepts that is vastly superior to anything a teacher might tell them. If a student doesn't understand the concept, an alert teacher or fellow student should have no difficulty noticing the student's frustration or lack of progress on a problem. This is in sharp contrast to a lecture, where a quiet student in the back row who has difficulty understanding the subject may not be discovered until a major exam.

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[2]Steen, L., (1983). Developing Mathematical Maturity. *The Future of College Mathematics* A. Ralston and G. Young (Eds.), Springer-Verlag, New York, 105

3.2.2   Cooperative Learning

Another popularcliche is that "Two heads are better than one!" More to the point, two people working together can bounce ideas off of each other, explain concepts that one student understands but the other does not, and in general provide mutual moral and academic support. [3]

During my HSSP class and especially during this unit, I discovered that groups were an essential part of the learning process. When one student has trouble with a concept or idea, the other can explain it. Having at least two people working with a computer encourages students to reason verbally. The teacher should encourage students to come up with a plan to solve their problems together. I found it helpful to announce a switch of "computer pilots" (those actually typing at the keyboard) approximately half way through each class period. When students worked alone, I noticed that they were much more frustrated and less creative than they were when they worked with a partner. On the other hand, groups of more than three students seemed to be rather crowded.

3.2.3   Learning in Context

People have a high level of intuition for many mathematical processes occurring all around them. Ask someone what happens when you drop a light, large object, such as a feather or a piece of paper, and they might say something similar to "it starts at a zero speed when you let go and increases, but quickly stops speeding up when the air catches it, then it falls at a constant, slow speed to the ground." This verbal description captures a complex pattern of aerodynamics, negative feedback, and a complicated differential equation in a sentence.

When learning about a discipline such as calculus, it helps if there is a concrete context to solve problems in. This way topics can be connected to a framework already held by the student. For example, rates of change can be introduced to velocity of a moving object. People learn new things best when they are taught in relation to other things they are familiar with.

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[3]Davidson, N. (1990). Small-Group Cooperative Learning in Mathematics. *Teaching and Learning Mathematics in the 1990s (1990 Yearbook)*, National Council of Teachers of Mathematics, Reston, VA, p. 52–61

An common debate concerns whether courses should teach from a concrete or theoretical viewpoint. However, these two perspectives are really complements of each other, not opposites, and both should be included. Theory is important. Through the learning of theoretical principles a more general understanding is gained. However, it is important to start with specific problems and then draw connections to these principles. [4]

3.2.4 Mathematical Reasoning

The field of mathematics can be thought of both as a set of rules with which people can solve problems, and the ability to know which rule to use in which problem.

Rules can always be found in books and tables. The ability to reason about which rule to use is harder to instill.

Calculus should be a course to help teach that kind of mathematical maturity. It is the last course in a sequence of algebra, geometry, and trigonometry. It is also the gateway to the mathematics of the infinite and the abstract. However, the reality can differ from the high-sounding philosophy. Many students come scarred out of calculus instead of enlightened.

> Many students– indeed, the vast majority– finish the calculus sequence unable to read or study mathematics, or to master on their own any part of the traditional core of university mathematics. Sometimes, more often than we care to admit students with special mathematical aptitude leave calculus with less mathematical maturity than when they entered. [5]

The course detailed in this paper tries to give students the power to reason about mathematical problems. Understanding the process of integration is more important than memorizing the rules of integration. Hence, very few rules are taught. Instead, experimentation, group work, writing, and discussion are emphasized. Computers, in particular, are a powerful tool with which to explore mathematics.

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[4]Janvier, C. (1990). Contextualization and Mathematics for All. *Teaching and Learning Mathematics in the 1990s (1990 Yearbook)*, National Council of Teachers of Mathematics, Reston, VA, p. 183–193

[5]Steen, L., (1983). Developing Mathematical Maturity. *The Future of College Mathematics* A. Ralston and G. Young (Eds.), Springer-Verlag, New York, 103

3.2.5 Appropriate Use of Technology

Computers and graphic calculators can add a tremendous amount of flexibility to the calculus student. The ability to move between multiple representations of functions can help students visualize important concepts. The computer can be a time-compressing device, performing large-scale computations in a short amount of time. Most importantly, technology's interactive nature can provide a laboratory where the student can experiment with numbers, equations, and mathematical processes.

It is important to realize that simply adding a computer to a calculus classroom may not necessarily improve a course. Software must be available, useful and easy to operate. Mathematical programs can provide a complexity of detail that can make it easy to lost sight of important concepts. On the opposite extreme, software can hide important calculations from the user, a feature that is often useful, but can be a hinderance if the goal is for the student to study the concepts the calculation is based upon. Finally, the need for courses to take into account the inherent interactiveness of the computer can not be stressed too much. Using a computer non-interactively is like using a tape recorder for a teacher– you can learn something, but it's pretty limited (not to mention dull).

3.3 Content Differences

This course is designed to cover most of the content of the second half of an introductory calculus class. However, there are some major differences in the emphasis and order of concepts taught. Instead of the teacher presenting a number of formal rules and integration methods, this course places emphasis on the students building upon and constructing their own intuition about how calculus works. Integration rules are learned in context as the students discovers a need for them.

One of the more noticeable differences in content is the fact that this introductory calculus course teaches integration without assuming knowledge of differentiation. This is done based upon an assumption of the author: namely, that integration describes everyday experiences and can be presented in an intuitive manner, building upon those experiences. Jay Forrester, founder of the field of System Dynamics, states this idea as follows:

In system dynamics, causality is implied by the structure of feedback loops. Causality is unidirectional from element to element around a loop. Causality is not ambiguous or reversible. From a system dynamics viewpoint, water flowing from a faucet causes the water level in a glass to rise. And the level of water in the glass can cause a person to shut off the faucet. But I have had people argue that no such causality is justified, that it is just as correct to assert that the water flows from the faucet because the water level in the glass is rising. I do not understand the basis for such a perspective, but confusion does seem to exist regarding causality and its relationship to modeling . . .

I believe that many students are confused, or made indifferent to, the direction of causality by being taught dynamics through the mathematics of differential equations. Focusing on flows as the mathematical derivative of a system state implies that the flow exists because the state is changing, rather than the state changes because of the flow. Nowhere does nature differentiate; in real systems, dynamic change arises only from accumulation, that is, integration. The common practice of considering differentiation and integration the inverse of one another tends to obscure the direction of causality. [6]

This course also broadens the definition of calculus to include topics traditionally taught in courses named "Differential Equations." In particular, in Unit 4, students spend time modeling feedback loops in first order "integral equations."

3.4   Course Divisions

The course is split into five units. They are:

1. Moving Objects

2. Fundamental Theorem of Calculus

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[6]Forrester, Jay W. 1980 "System Dynamics – Future Opportunities." In A.A. Legasto Jr. et al (ed.) *System Dynamics*. Series: TIMS Studies in the Management Sciences, pp. 7-21. New York: North Holland. Also D-3108-1, MIT System Dynamics Group

3. Integrating across Space

4. Discovering Dynamic Systems

3.4.1 Moving Objects

The overriding theme of calculus is the relationship between rates of change and the accumulation of those changes. The first unit of this course is intended to promote student inquiry and thought on the most visible manifestation of this concept: moving objects. This unit allows students to design experiments to answer questions about the relationships between the acceleration, velocity, and position of a falling weight or a rolling marble. Little calculus notation or terminology is used, but the ideas introduced in this lesson provide a foundation for much of the rest of the course.

3.4.2 Fundamental Theorem of Calculus

Calculus can be seen as the gateway between the largely discrete mathematics of the pre-college years and the continuous mathematics of engineering and science. Unfortunately, students are often abruptly thrust from one domain to the other with the introduction of the limit, a concept that is mathematically appealing but often confusing, particularly when introduced before students have had much experience or practice working with continuous systems.

This unit strives to achieve a gentler and more cognitively appropriate transition. Through discussion, students develop methods of numerical integration, and then apply them using computer based spreadsheets. After some initial spreadsheet verification of the experiments they performed in the previous unit, they begin to search for improvements to this method of integration. Students work in small groups and design new methods of integration (such as the trapezoidal rule), explore the relationship between size of dt (time interval) and the error in their integrations, and search for general rules to integrate with (such as the power rule). The unit ends with group presentations and summaries of the rules and idea learned. Through discovery and problem solving this unit uses numerical integration as a step into the symbolic calculus.

3.4.3 Integrating Across Space

The primary applications of integration fall into two primary categories: integration of dynamic systems, where the focus is the summation of changes of quantities over time, and spatial integration, where the focus is on the summation of changes of area, volume, density, or electric charge, through space. All of the examples in the previous units have been of moving objects and have involved integration over time. Here, the students solve problems involving area and volume as they learn the connections between adding up areas and adding up dynamic changes.

3.4.4 Discovering Dynamic Systems

Traditional calculus courses often ignore many topics that might fall under the domain of a course entitled "Differential Equations." Feedback loops, exponential growth, and logistical growth are all treated cursorily or not at all in most calculus textbooks. However, these mathematical ideas describe important processes ranging from population growth to the operation of a thermostat. Part of the lack of discussion of many of these concepts in calculus is due to the orientation of calculus towards the differential point of view, rather than the integral. Solving these systems using the methods of differential equations may be complicated, but integration of the systems with a computer can make analysis much easier and more concrete

In this unit, students explore dynamics systems using spreadsheets and simulation programs. They examine and model real-world processes such as the examples mentioned above. Using the tools of calculus, they attempt to understand how their dynamic world works.

3.5 Course Objectives

At the end of the course the primary goal will be for students to know when, and where to use integration to solve problems. In particular, they should know how to model problems, that is, how to describe them with integrals or integral equations in a manner that will lead to a solution. They will learn some of the ways of solving the integrals, but the emphasis will be on modeling.

Secondary goals will be for students to distinguish problems that involve a dynam-

ical system from a problem that needs integration over space. For dynamical systems, they should be able to see them from a causal perspective, and model them with equations or the aid of a dynamic modeling and simulation package such as STELLA. For spatial systems, they should be able to identify a "sample slice" to be added up, and then set up the integral. Finally, students should be able to do qualitative, graphical integration, do numerical integration using a calculator or spreadsheet and to calculate simple symbolic integrals.

Chapter 4

The Fundamental Theorem of Calculus

4.1 Unit Overview

4.1.1 The Fundamental Theorem of Calculus

Loosely speaking, the Fundamental Theorem of Calculus states that if a function is the rate of change of a quantity, the accumulation, or integral, of that change is equal to the area under the function. Since Newton discovered this in the seventeenth century this theorem [1] has been the heart of calculus. In this unit, students will study the accumulation of dynamic change through numerical integration. At the end of the unit they will attempt different methods of making a more accurate form of integration, leading to a discussion of limits. In the unit after this one, they will work more with connections between integration and area.

4.1.2 Unit Objectives

The goals of this unit are:

1. To teach students methods of calculating integrals or accumulations of change.

2. To have students discover a need for a symbolic set of integration rules.

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[1] Actually, two theorems

3. To let students discover the concept of a limit through trying to perform "accurate" calculations.

4.1.3 Divisions of Unit

This unit has three parts. They are:

1. Discussion: What is Integration? (I)

2. Application: Integrating Specific Problems

3. Application: Discovering Integration

4. Discussion: What is Integration? (II)

Each of these three parts serves a specific purpose. Discussion I introduces integration, and helps students to verbalize their intuition regarding the effects of change. It also presents an approximate method of calculating the integral. Next, students practice this method of integration with exercises set in a science fiction future. After that, students work on a series of exercises designed to introduce symbolic integration and the limit concept in the context of what students already know about numeric integration. Finally, students sum up what they have discovered in discussion II.

4.1.4 Using Spreadsheets For Numerical Integration

Most of this unit involves students working in pairs or triples using a spreadsheet program to perform calculations and create graphs. The role of the spreadsheet is to allow the student to explicitly perform the iterative calculations that numerical integration involves. While graphing calculators or other types of computer programs might be used as well, spreadsheets have many advantages. Students can enter a formula once, and then quickly create many similar formulas. Nothing is hidden in a spreadsheet – once a formula is entered it can easily be viewed or modified. A modification makes all other formulas based on that changed cell change as well. Spreadsheets make it easy to organize numerical information. Spreadsheets also act as "time-compressors" – the fast speed at which calculations can be entered, calculated, graphed, modified, and graphed again encourages experimentation and discovery. Finally, a spreadsheet

	1	2	3	4	5
A	Time				
B	0.0	0.5	1.0	1.5	2.0
C					
D	Input	Rate	Function		
E	30.0	35.0	40.0	45.0	50.0
F					
G	Dt				
H		0.5	0.5	0.5	0.5
I					
J	Integral				
K	0.0	15.0	32.5	52.5	77.5

Figure 4-1: A typical spreadsheet integrating a function

allows the student to view their calculations in several different manners: as formulas that they have entered, as a table of calculated numbers, or as a graph of a function.

Spreadsheets are generally available and relatively easy to use. In my HSSP class, we used Microsoft Excel, a popular program that is in common use. The only major problem that I found students had with the software was that the method of graphing in Excel is somewhat cumbersome and non-intuitive.

More sophisticated programs such as Mathematica are available, but are expensive and overly complicated for the purposes described here. A few programs (in particular, a newly released program called Function Probe) seem designed for this type of flexibility and discovery-oriented use. Function Probe, for example, is designed around the concept of multiple representations of functions. Using this program, the student can easily enter iterative equations, create a table of numbers, graph the results, and turn the graphs back into a table of numbers. [2] As computers become widely available in the calculus classroom such programs are sure to proliferate.

4.2   Discussion: What is Integration? (I)

This unit begins with a teacher led discussion. It has three principle goals.

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[2]Confrey, J. and Erick Smith, Student Centered Design for Educational Software, *Proceedings of the Conference on Technology in Collegiate Mathematics: The Twilight of the Pencil and Paper.* Addison-Wesley Publishing Co. Reading, MA. 133-137.

1. To make explicit students' intuition about the relationship between acceleration, velocity, and position.

2. To generalize that intuition to cover other systems of systems of dynamic relationships.

3. To cooperatively develop a method for calculating an approximate integral of an arbitrary rate.

4. To introduce standard notation for rates of change and their integral.

In my description of this discussion in the course such as the one following, I will intersperse direct descriptions of the course with many short dialogues. These are not transcriptions of exact conversations I had in my class (although many are similar) nor are they intended to dictate how others should lead their discussions.

The purpose of these fictional interactions is to illustrate a typical class, give more realism to the description, and to show issues that students and teachers might encounter. There are many students in this class, but I will refer to them only as "Student." For no particular reason, I will call my mythical instructor "Professor Jones."

Professor Jones begins his class by writing on the board "What happens when an object moves?" He then asks his class to respond to this seemingly obvious question.

Student: Umm, what do you mean?

Professor Jones: Well, when I toss a ball, like this (tosses a ball), what happens?

Student: You mean that it goes from place to place?

Professor Jones: Good! (writes in the corner of the board, "changes position")

Student: How about that the ball hits something?

Professor Jones: Ok, but let's just talk about how it moves.

Student: Well, the ball has a certain velocity...

Student: Oh, just call it speed!

Professor Jones: I'll write both. What does the speed of the ball have to do with the change in position?

Student: Well, a faster speed will make it go farther in the same time. A slower speed will go less.

Professor Jones: I'll draw an arrow from speed to position.

Student: And acceleration has something to do with that, doesn't it?

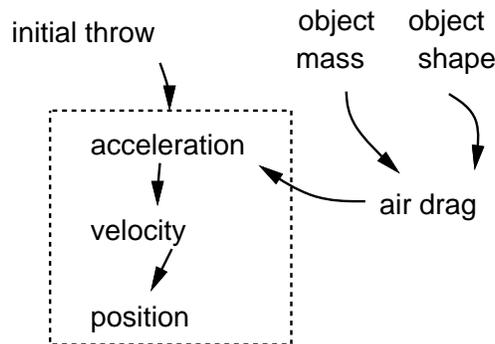*Student:* Acceleration changes the speed.

Professor Jones: (writes acceleration on the board, connects it to speed). What else?

Student: What about the drag from the air? That slows the ball down!

Student: Wait... Isn't that really just a kind of acceleration?

Professor Jones: Good for you. (writes air drag on the board and draws an
                          arrow to acceleration)

Professor Jones continues the brainstorming session until the class has filled the board with a web of connections. Then he draws a square around position, velocity and acceleration.


Professor Jones: All of these factors are important, but the relationship
                          between these three is key. In particular, acceleration
                          changes the speed (or velocity, as it is often called). Think
                          of acceleration as giving gas to a car. Press the gas pedal
                          down, and the speed of the car keeps increasing. The
                          speed of the car is both the measure and the cause of
                          changing distance. This is called integration, when one
                          thing causes a change over time of another thing. This is
                          what a lot of calculus is all about.

4.2.1   Notation

At this point, Professor Jones introduces some notation into the discussion. The notation gives students a vehicle to use when referring to an integral. It is not important that the students have a complete understanding of the full meaning of the symbols involved (that will come later). As they learn more calculus, they will build their own knowledge of the mathematical language that supports the subject.

(Professor Jones writes the following diagram on the blackboard)
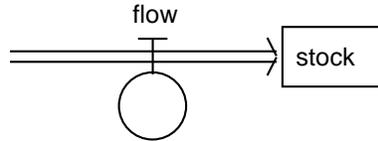
Professor Jones: These symbols are a way of describing what we have been talking about. The square is a stock – something that accumulates in a system. The double arrows are the rate of change of a stock. An example of a stock that we encounter all the time is water in a bathtub. Water accumulates in the bathtub because of...

Student: A faucet!

Professor Jones: Good. A faucet is an *inflow* into the bathtub. What else causes a change in the amount of water?

Student: Yeah, yeah, a drain, of course.

Professor Jones: Right. Notice that a stock can not change by itself. It has to be raised or lowered by a flow.

So what does this have to do with a moving object? Well, if I drive a car at 60 miles per hour, what is happening?

Student: Ah, I get it! The location of an object is the stock – my speed changes it and must be the flow. And I can't instantly beam from one place to another – I've got to change my speed.

Professor Jones: And what controls the speed? For example, if you are driving down the road and decide to go faster, what increases the cars speed?

Student: The gas pedal?

Professor Jones: Right, but what does pressing the gas do? The answer is that it causes an acceleration. Velocity is both a stock *and* a flow in different situations. What other stocks and flow combinations can you think of?

Students call out pairs such as water/water flow, money in a bank/interest, and library books/check-outs. Professor Jones keeps a list on the board.

Professor Jones: Good! Here is a more technical way of writing the relationship between position and velocity.

Professor Jones then writes the following equations.

$$p(t) = \int v(t)dt$$

Professor Jones: This is a more precise way of writing an equation for the principle behind what we have been talking about. Don't worry what it all means, we'll discuss it in detail later. Essentially the equation says that the function p(t), the position, is the integral of v(t), the velocity. The important thing is the tall thin "s". That means integrate the function v(t). The "dt" means integrate the changes over t, the variable for time. The whole thing is read, "The integral of v of t with respect to time."

Student: But how do you know how far you've gone?

Professor Jones: We add boundaries to our integral like this (writes new equation). This is read, the integral of v of t from t equals 0 to t equals 3. In other words, ho far have we traveled from 0 seconds to 3 seconds.

$$p(t) = \int_0^3 v(t)dt$$

39

4.2.2 Calculating position from a constant velocity

Professor Jones would like to lead the class into developing a method for calculating the position from a rate. He starts off with what they know – how to calculate position with a constant velocity.
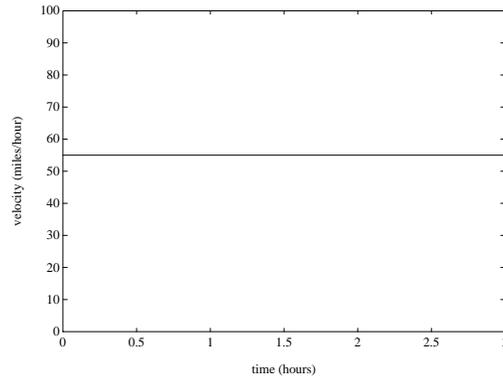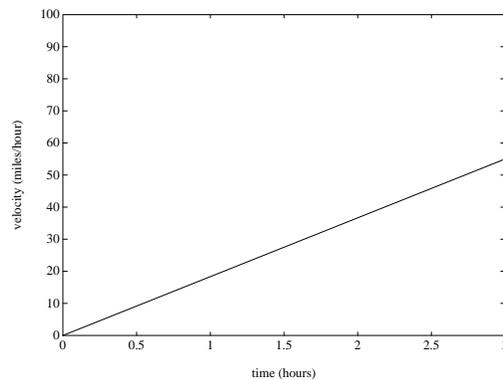

$$v(t) = 55$$

$$p(t) = \int_0^3 55\,dt$$

(Professor Jones draws the above graph and equation on the chalkboard)

Professor Jones: Let's review some stuff we should all know. This is a graph of me driving down the local freeway for three hours at 55 miles per hour. How far will I go?

Student: D equals R times T, right? So, you've driven 165 miles.
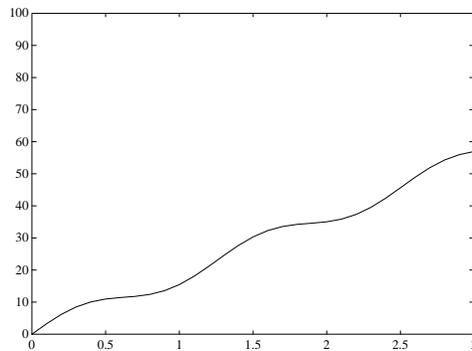
$$v(t) = \frac{55t}{3}$$

$$p(t) = \int_0^3 \frac{55t}{3}dt$$

Professor Jones: How about if my velocity is the above graph? I start going very slowly, and increase my speed until I am going 55 miles per hour at 3 hours. How far would I have gone?

Student: Well, couldn't you take the average or something? It shouldn't be hard...

Professor Jones: Well, maybe that would work and maybe not. What if my speed varied even more – if I sped up, then slowed down like this? (draws new graph on blackboard) We need a general method.

### 4.2.3 Calculating position from a changing velocity

The stage is now set for Professor Jones to introduce numerical integration. First, he presents the general concept of approximating integrals with constant rates, then he goes through the algorithm step by step.

Professor Jones: Let's review what we know. We can find how far we have traveled as long as our speed has been constant.

Well suppose, my speed is this (draws graph). I travel for 2 hours at 5 miles per hour, then 4 hours at the slower speed of 3 miles per hour. How far have I traveled?

Student: Umm, 2 times 5 is ten, plus 4 times 3 is 12 plus 10 is 22. You've gone 22 miles.

Professor Jones: Right, you can just add them together. We can do approximate integrations of any graph just by making approximations of the rate, in other words, turning the varying graph into a series of constant graphs. Here, we turn a increasing speed into a series of constant speeds. (draws graphs on blackboard)
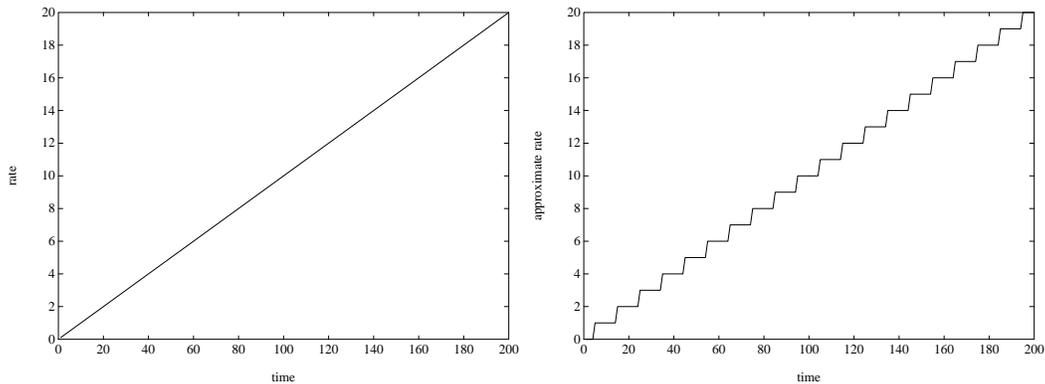
At this point, Professor Jones leads the class in developing this algorithm for numerical integration.

1. Break the function into a set of discrete intervals.

2. For each interval, assign the function a *constant* value. The easiest way to do this is use the value of the *beginning* of the interval.

3. Starting at the first interval, go step by step through all the intervals, calculating the influence of the constant rate (for that interval) on the integral.

4. If these are the relevant variables:

$$
\begin{aligned}
d_{new} &= \text{The value of the integral at the end of the interval} \\
d_{old} &= \text{The value of the integral at the beginning of the interval} \\
v &= \text{The constant rate of change for that interval} \\
dt &= \text{the change in time over the interval}
\end{aligned}
$$

Each interval, you need to calculate the new value of the integral (d_{new}) based upon the old value (d_{old}) plus the change in the integral.

$$d_{new} = d_{old} + v(t) \cdot dt$$

At the end of the discussion, Professor Jones gives his students a worksheet covering stocks, flows, and integration. An example of such a worksheet is *Problem Set 1* in the appendix. It includes a question asking for a paragraph of writing about stocks and flows. Professor Jones thinks that this might be a good way to get his students thinking about the relationships between rates of change and integrals.

4.3   Application: Integrating Specific Problems

Professor Jones begins his next class with a review sheet to remind his students of the rules of integration. A copy, marked *Methods of Integration Review Sheet*, is in the appendix. Jones lets his students work in pairs to fill in the blanks using their notes from the previous class, then goes through it with them. By having them briefly review both in pairs and in a group, he ensures that everyone is prepared for the day's problem solving activities.

The last section of the review sheet covers using a spreadsheet to do numerical integration, something Jones and his class have not yet discussed. Jones takes extra care to make sure everybody understands the concept of a spreadsheet, then demonstrates

how one works using an overhead projector hooked up to his computer.

After this introduction, the class is ready to work on the exercises. Professor Jones splits his class into groups with two or three members each and assigns each group a computer.

The exercises, (entitled *Integration Problem-Solving Exercises 1* in the appendix), are designed to give students practice in calculating numeric integrals. The questions all concern acceleration, velocity, and position.

Professor Jones tells his students they must do the first integration by hand, making tables of number with a calculator. He would like them to get a "feel" for the integration before they have the computer do it all. He also gives them guidelines for how small their time interval, dt, should be, although he lets them choose its exact size.

While the students are working, Professor Jones keeps an active eye on the progress of his students. He pauses next to one pair of students who have raced through the calculator integration, and tells them to start working with a spreadsheet. He notices another student working mainly on his own, with a partner who is staring out in the air. Professor Jones finds out that the partner does not understand the algorithm, and gets the first partner to explain it to the second. The groups all work at slightly different paces, according to their abilities, understanding, and computer experience.

4.4   Discovering Integration

As Professor Jones notices students finishing the first round of exercises, he gives them a second set, entitled *Integration Problem-Solving Exercises 2*. Students need only do one problem out of three. Again, they work cooperatively in pairs or trios to solve the exercises.

These problems are intended to introduce students to more general aspects of integration. In particular, the assignment asks broad and open-ended questions about integration as a process. To answer these questions students need to understand the nature of integration, and consider how it can be improved.

The first question asks students to find rules of integration for polynomials. Students should already know how to find the integration of a constant rate. Finding

the a general rule for integrating constantly increasing rates is not easy. Finding a rule for quadratic rates is a challenge. Professor Jones is especially careful to suggest that groups solving this problem use a very low time interval, or the problem becomes almost impossible.

The second question asks students to explore other methods of numerical integration. The trapezoidal method is not difficult to figure out in concept or in detail. If students have trouble, Professor Jones asks them questions to be sure they understand how the class algorithm (rectangles) worked.

The final question leads students to explore the limit concept in search of a more exact method of integration. Students test integrals with smaller and smaller intervals, and graph the resulting errors. As part of this problem, students need to define the concept of error, and determine how to measure it.

4.5   Discussion: What is Integration? (II)

After most of the students have finished their assignments and reached a conclusion, the last discussion can begin. This is an important point in the class. Here students have an opportunity both to reflect on lessons learned, and share those lessons with their classmates.

Professor Jones gives each group about five minutes to make a presentation on what they discovered, complete with graphs and diagrams. After the presentations, Jones sums up what each group accomplished.

Most of the students should now be comfortable with the concept of integration being the accumulation of change. While all of their experience has been with numeric integration, they have investigated questions that directly relate to symbolic integration and the Fundamental Theorem of Calculus. Professor Jones starts the last phase of the discussion from there.

Professor Jones: Today, you looked at what mathematicians call *numeric integration*, that is the calculation of approximate integrals. Any ideas how we could get *exact* integral?

Student: What if we used trapezoids or triangles instead of rectangles, like I did with my homework?

Professor Jones: Well, that would be better, but would it be exact?

Student: No, but couldn't we use really small intervals? That would be even closer.

Professor Jones: Again, it would be close, but not exact. Did anyone notice how the error changed as the intervals got smaller and smaller?

Student: On my graph, the error came closer and closer to zero, but changed less each time. It never reached it though (as far as I could tell).

Professor Jones: Ok. Well, let us suppose that somehow, we had infinitely small time intervals, and got that error down to zero. Then, we would have our exact integral. The integration that is most often seen in calculus is founded on the idea of those infinitely small time intervals. Too bad we can't set our spreadsheet to calculate infinitely small *dt*'s!

Student: But, if you can't calculate it, what good is it?

Professor Jones: I didn't say that you couldn't calculate it, only that you couldn't calculate it numerically. Anyone else want to help answer that question?

Student: Well, in my homework, I discovered a rule for integrating. If you have a function that is some constant number times t, you can get the integral by plugging in the same value of t into the equation t squared times the constant divided by two. There must be other rules, too.

Professor Jones: Very good. So good, in fact, that you just took my whole lecture away from me. That's all there is to calculus, a set of rules to do integration by. In fact, there are so many of these rules, people write books just listing them. But the rules aren't as important, what's important is knowing how to use the rules to solve problems.

Professor Jones ends the discussion by giving examples of integration rules such as the power rule and some of the rules for trig functions. He leaves them with a list of rules to memorize, and a set of rather standard (and boring, think the class) problems for them to complete.

The computer exercises brought home the idea of an integral, long before Professor Jones discussed symbolic integration or handed out the lists of rules for integrating functions. Students had all the background to understand a discussion on the Fundamental Theorem of Calculus. Just as importantly, they could contribute ideas gained during the exercises to the discussion, helping them to build an understanding of symbolic integration that complemented the understanding gained from the work they had done.

Appendix A

Handouts for HSSP Course: "Calculus Can Be Fun!"

Calculus Can Be Fun!
A Hands-On Introduction to Integration

Course Description

Math does not have to be dry, boring or seemingly useless. Math doesn't necessarily involve lots of incomprehensible symbols, either. Instead, the processes and concepts that make up mathematics are the glue that hold our universe together. Almost everything we do involves math, even something as simple as operating a tap to fill a glass of water!

This course explores the topic of integration, traditionally taught in the second half of an introductory calculus class. Through problem sets, discussion, lab work, and computer experiments, we will investigate the principles behind integration, and how they describe fundamental processes behind many everyday experiences.

Note: This class *does not* follow the AP Calculus syllabus.

Prerequisites: A background in algebra is needed. Experience with trigonometry is preferred, but not required. Students with an extensive background (one year or more) in calculus will not be allowed to enroll. No computer experience is necessary.

Enrollment limited to 10 students.

Calculus Can be Fun:

A Hands-on Introduction to Integration

Course Syllabus

William Glass, November 30, 1992

Day 1: Acceleration/Velocity/Position

    A) Lecture: Introduction of Course (very brief)

    B) Discussion: Modeling a falling object (I)

- characteristics of object
- Draw up list of questions about a/v/p

    C) Lab: Measuring and analyzing falling objects

- down staircase (1 group)
- falling weight tied to ticker tape (2 groups)

    D) Discussion: Modeling a falling object (II)

- results of lab
- relationships
- stock/flow
- equations (integrals)

    Handouts: A/V/P Lab Task Sheet "What is Integration"

    Homework:

- Lab Wrap-up (split up among group)
  - organization of graphs and data
  - short answer questions
- Graphical Integration of a/v/p (motorcycle)

Day 2: Numerical Integration

    A) Discussion: "How could a computer have helped solve yesterdays problem?"

B) Computer: Numerical integration with spreadsheet

- falling object
- motorcycle from homework

C) Computer centered discussion Collaborative building of STELLA model on overhead projecting macintosh

Handouts:

- "A Quick and Dirty Guide to Calculating with a Spreadsheet"
- "Numerical Integration Worksheet"
- "Numerical Integration Algorithms"

Homework: Discovering Integral Formulas (needs calculator)

Day 3: Connecting Symbolic Integration to Numerical

A) Lecture: Fundamental Theorem of Calculus

B) Lecture: Integration methods: the Power rule, trig functions

Handouts: Practice Integrals and solutions

Day 4: Lecture: Integration methods: U-substitution

Handouts: Practice Integrals and Solutions

Day 5: Integrating over space

A) Discussion:

1. "What is the area under a sine wave?"
2. "What is the area between a sine wave and a cosine wave?"
3. "What is the formula for the area of a triangle?"

B) Small group problem solving: Find volume of strange looking solid

C) Discussion: Groups present solids and solutions

Homework: Strange looking solids filling up with water

Day 6: **A) Lecture:** Tools for Integration

- integration methods

- tables

- maple

- STELLA

B) Group Problem Solving

Day 7: Group Problem Solving and Modeling

Groups will split up and do one of:

Lab:

- Modeling an RC circuit – When is a TV tube safe to touch?

- Cascading buckets of water – How to make bottom bucket level constant by controlling top flow?.

- Other labs to be decided on

Handout:

- Basic Circuit Facts (KCL, KVL, $V = IR$, $Q = CV$, $i = \frac{dq}{dt}$)

- Other handouts as appropriate

Assign: Final Project

Day 8: Feedback Loops / Logistic Growth

A) Lab: Play epidemic game

B) Discussion: Results of game

Day 9: Feedback Loops / Logistic Growth

Computer: Modeling an Epidemic

Due: Final project

Day 10: "Uncensored Calculus"

Lecture: Other neat stuff: multiple integrals, finding max and min of functions and drawing them, etc.

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HSSP Syllabus
revised

As most of you have noticed, we have been going considerably slower than the originally proposed syllabus. Teacher and student absences and my underestimation of the length of the experiments and computer work have caused a need for an adjustment of this term's schedule.

Here is a list of the topics for the rest of the term. If you plan on missing any classes, please try and let me know before hand. My address/phone number is at the bottom of the page.

Week 6: 4/11/92 Numerical Integration

Week 7: 4/18/92 Symbolic Integration: Limits, Power Rule, Trig Rule

Week 8: 4/25/92 Problem Solving with Integration (Area/Volume)

Week 9: 5/2/92 Logistical Growth: Exploring Epidemics

Week 10: 5/9/92 Computer Modeling and Integration

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What is Integration?
A Quick and Dirty Guide

Accumulation

Simply put, integration is a mathematical way to describe the process of *accumulation*, or the adding up of changes of a quantity. As you might guess, this is pretty important, and a lot of elaborate mathematical equations and formulas have been created to help understand and solve problems that involve this. Integration is one half of the field of *calculus*, the mathematics of change. The other half is called *differentiation* and is the calculating of rates of change.

You don't have to use any fancy formulas to understand what integration is. Whenever you fill a bathtub, you are controlling the flow of water through the faucet causing a change in the amount of water in the tub. You may be only be thinking "If I let the water run full blast for a few minutes, I can fill the tub." However, that's the same idea behind saying, "If I let 25 liters/minute of water flow out the faucet for two minutes I will have a 50 liters of water in the tub."

Another everyday example of integration you may be familiar with is your bank account. What causes change in a savings account?

Accumulation may not always be a process that is wanted. For example, the U.S. Government is approximately $\$4,000,000,000,000$ in debt. The government finances its debt by borrowing money from investors, then repaying them with interest. If the average interest rate is 5%, then this means the government is paying $\$200,000,000,000$ in *interest* every year, adding to the total debt. This is not even counting the new debt that keeps accumulating due to budget deficits. Integrate that (and then show it to your congressman)!

In the engineering and scientific world, integrals are used for applications as varied as calculating the average density of an object (by adding up the densities of all of the parts), finding the ability of a capacitor to store electrical charge (adding the ability of each section to store charge), and tracking the motion of the planets (by repeatedly calculating incremental changes to positions and adding them to gain new positions).

Types of Integration

Problems that can be solved by integration fall into two categories: problems that involve change over time and problems that involve change over an area or space. Examples of the first kind are called *dynamic systems*, examples of the second kind are called *spatial systems*. We refer to the formulas used to describe them as *integration over time* and *integration over space*. For now, we are going to concentrate on dynamic systems (we'll talk more about spatial systems later in the course).

A Moving Car

An example of a dynamic system would be a moving car. The car has a certain velocity that causes changes in the car's position. To find out the position of the car at any point in time we will have to add up all of the changes to the car's position from the velocity.

To see how integration works, let's look at the following example:

Suppose Stephanie drove her new car straight north from her house starting at 12:00 noon. If she drove for one hour at a velocity of 50 $\frac{miles}{hr}$, two hours at 25 $\frac{miles}{hr}$ and one hour at 35 $\frac{miles}{hr}$, show graphically how far Stephanie had driven at any point in time. How far had Stephanie driven at the end of the four hours?
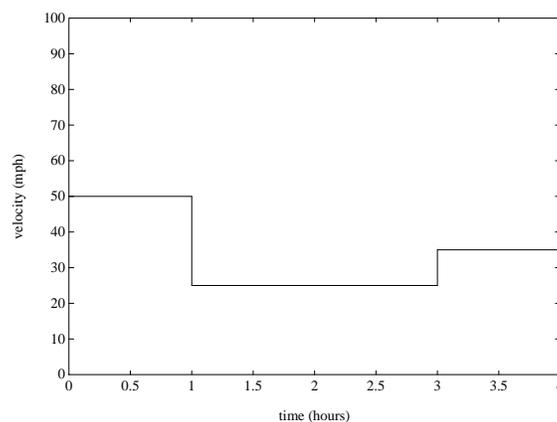

Figure A-1: Graph of the velocity of Stephanie's car over the four hours she was driving. What was her position at any point in time?

To draw a graph of the position of the car, we integrate the car's velocity over time. In other words, we add up changes in the position of the car caused by the speed.

In the first hour, Stephanie will have traveled:

$$1hr * 50\frac{miles}{hr} = 50miles$$

In the next two hours Stephanie travels:

$$2hr * 25\frac{miles}{hr} = 50miles$$

So far she has traveled:

$$50miles + 50miles = 100miles$$

In the last hour she traveled:

$$1hr * 35\frac{miles}{hr} = 35miles$$

So her final distance is:

$$100miles + 35miles = 135miles$$
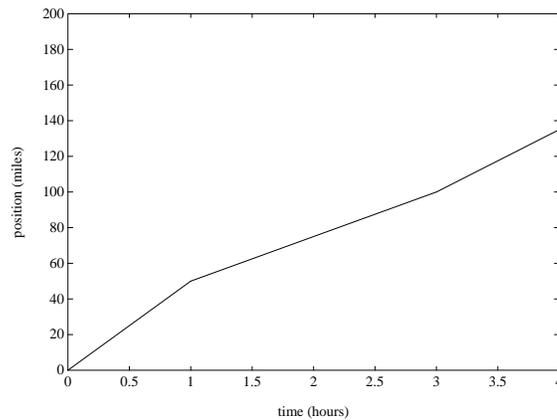
Figure A-2: Graph of the distance of Stephanie from her house

Describing Dynamic Systems: Stocks and flows

Sometimes it is handy to be able to able to show dynamics systems visually. To do this we will use two kinds of objects, *stocks* and *flows*.

A stock is something that accumulates or deccumulates according to a rate of flow. The classic example of a stock is a bathtub full of water. The bathtub has an inflow (the faucet) that lets in water at a certain rate. It also has an outflow (the drain) that releases water at a different rate.

For example, in the preceding example, the distance of Stephanie from her house went up or down, based on her velocity away from her house. Stephanie's position is the stock, and her velocity is the associated flow.

Stocks never can change directly. They always have to be changed by a flow. Water in a bathtub can not instantly vanish, it has to flow out of the tub with a certain rate of exit. Nor can Stephanie instantly teleport from one location to another – she has to move in one direction or the other with a certain speed.
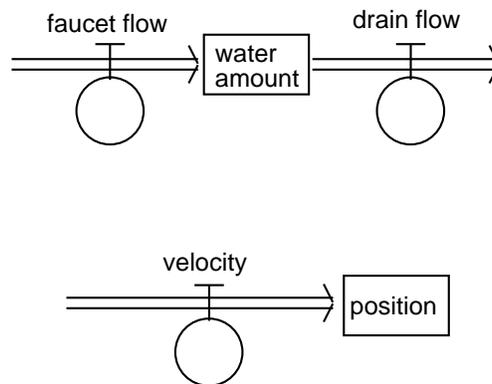

Figure A-3: Two stock and flow combinations

We will use this notation from time to time. In particular, we will use a computer modeling program STELLA that is based upon the notion of stocks and flows.

Describing Dynamic Systems: Integrals

Formal mathematics uses *integrals* to symbolically describe integration. If $f(t)$ is a function describing the rate of change of a quantity, then:

$$\int f(t)dt$$

describes the integration of that function over time. You can pronounce this "The integral of f-of-t with respect to t". This simply means that y is the result of adding up the function $f(t)$ as t (time) increases. Y is the equivalent of a stock, and $f(t)$ is the equivalent of the flow. We'll talk a lot more about integrals later in the term, starting around week 3.

Conclusion

This handout isn't meant to teach you everything there is to know about integration – far from it. Instead, it has tried to give you an high overview of the concepts and notation. We'll spend lots of time talking about and using integration this term, so you'll have plenty of time to learn more.

Problem Set #1

This is the first of several problem sets you will receive in this class. In general, many problems will be easy, some will be hard. You should try all of them, if you get stuck, ask for help.

1. Are the following stocks (things that accumulate), or flows (the rate at which things accumulate)? Give the units that each can be measured in.

Variable	Type	Units
Bank Savings	*stock*	*dollars*
Sales of a product		
Calculus Students		
Salary		
Atmospheric pollution		
Position		
Velocity		
Acceleration		

2. What are some possible flows for the stocks below? What are the units that the stock and the flow can be measured in?

Stock	Flow	Units of Stock	Units of Flow
Population	*births, deaths*	*people*	*people/year*
Computers in a store			
Nuclear Weapons			
Books in a Library			
Trees			
Air in a balloon			

3. Up until now we have focused on a very specific example of calculus: the mathematical descriptions of moving objects. Yet change occurs all around us, every day. And with it comes a need to describe it. Even if we don't know exactly what the specifics are, we know that calculus (and hence integration) can be used to describe and analyze change and it's effect. Your assignment is to write a paragraph on some system that displays change. Specifically, you should:

   (a) Identify the stock in the system.

   (b) Identify the flows that change the stock.

   (c) Draw the stock and the flow.

   (d) Mention the units of the stocks and the flows.

   (e) Discuss how you would measure changes in the stock and the flow.

   (f) Give a rough idea of what the flows are (constant, increasing, decreasing, etc.).

   (g) Give a rough idea of what the value of the stock is over time.

   An example paragraph appears below:

   > An important system that is continually changing is the population of the United States. The population is the stock, and is increased by inflows of births, and immigration. Population is decreased by outflows of deaths and emigration. Population is measured in people, and the flows can be measured in people/year. Measurements of the U.S. population is done every ten years through a census. The population of the U.S. is about 260 million. The birth rate is increasing exponentially, and consequently, so is the population.
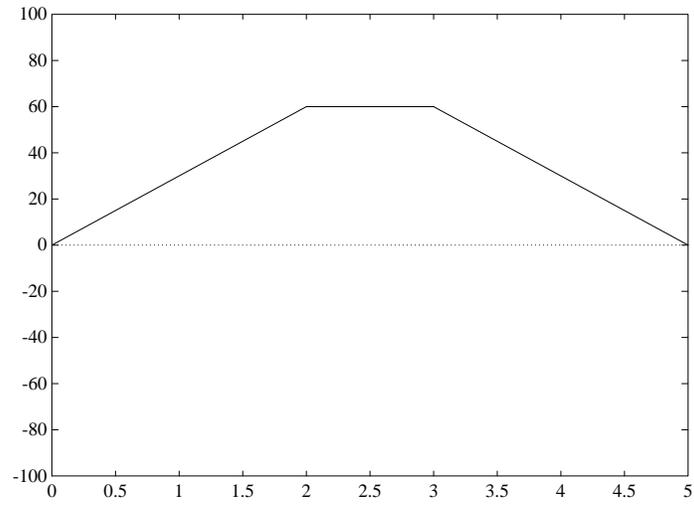
4. Celeste has a motorcycle that she likes to ride. On Friday, she hopped on her motorcycle and drove due *West*. For each of the following graphs that show possible velocities she drove at over the several hours she was riding, draw her matching position. (remember, a negative velocity means that Celeste was heading *East*, back towards her starting point)
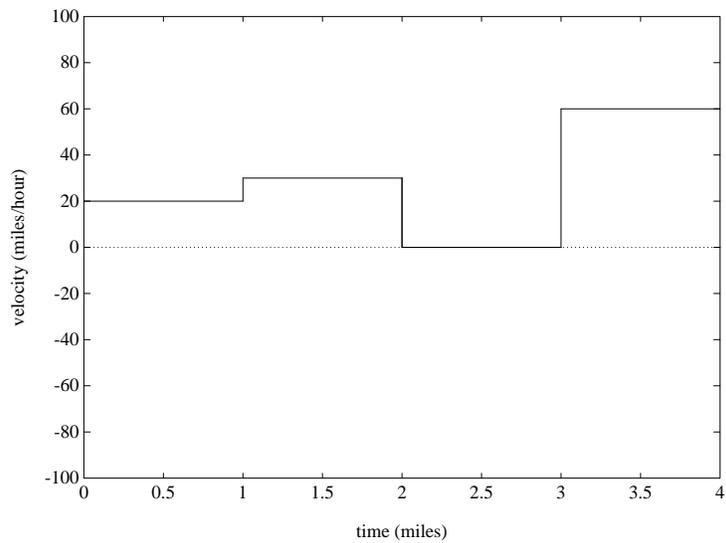

(a)
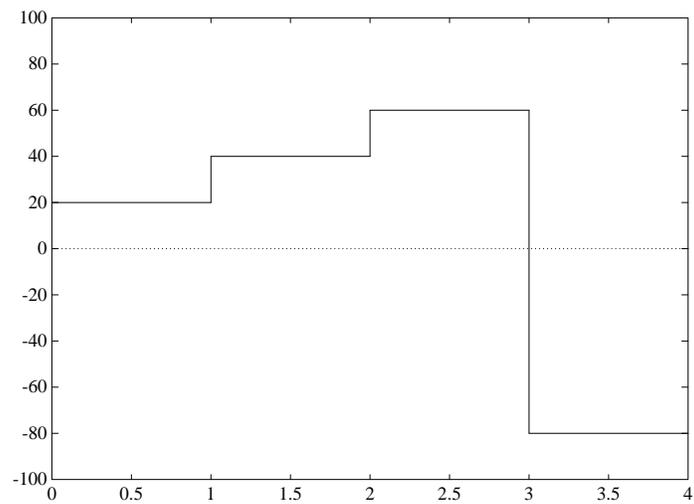
(b)

(c)

5. Last summer, Chris and some friends went on a driving tour of the United States. Over the course of the summer, they drove approximately 3000 miles. At the end of the summer, they drew up respective maps of how far they had traveled every day. Then, they wondered how far each of them had averaged traveling per day. (Not all of Chris' friends had been with him for the entire trip) For each graph of the distance traveled below, draw a corresponding graph of how fast that person traveled.
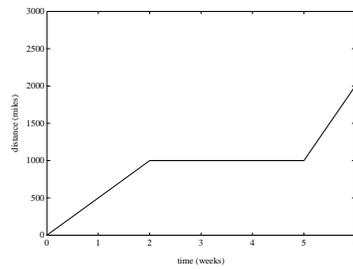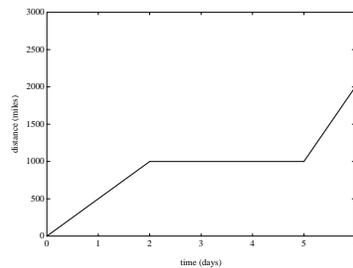
(a)
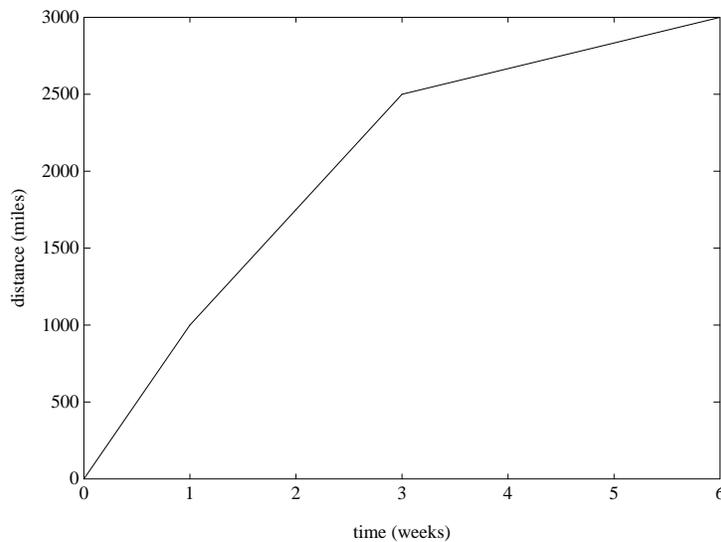
(b)

Methods of Integration I: *Numerical Integration*
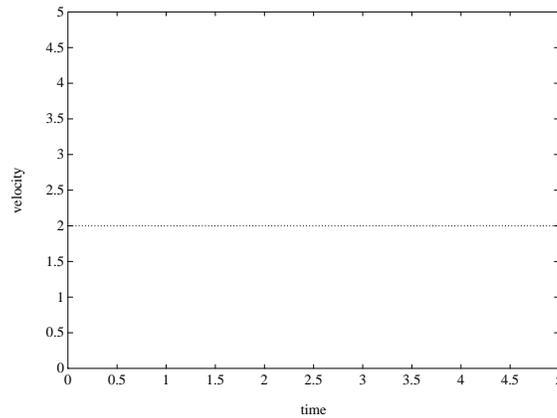
Review Sheet

I. Integrating constant rates

Figure A-4: Car with constant velocity of 2 for 5 seconds.

A quantity that is changing at a constant rate for a set amount of time may be integrated with the formula:

In Figure 1, the distance the car moves is:

II. Integrating arbitrary rates

Integration rules for special functions are useful integrating tools, but a method is also needed to integrate *arbitrary* rates. One way to do this is to *approximate* the function with a series of constant functions.
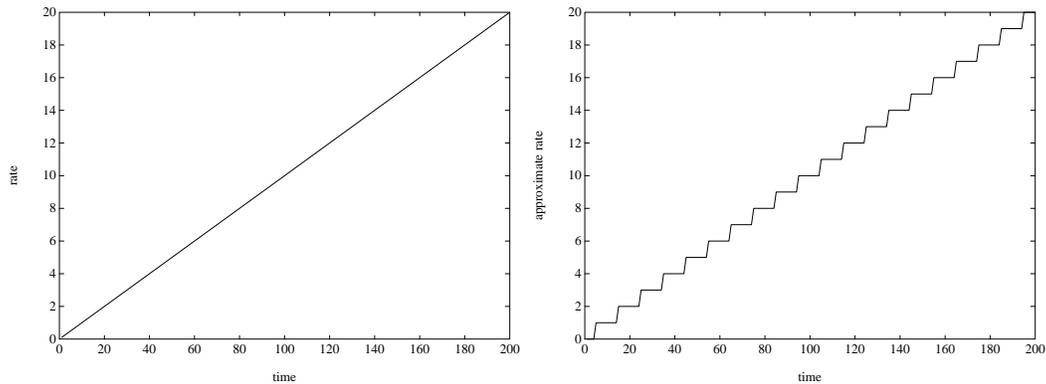

Figure A-5: Transforming an arbitrary function into a series of constant functions

To integrate an arbitrary function: [1]

1. Break the function into a set of discrete intervals.

2. For each interval, assign the function a *constant* value. The easiest way to do this is use the value of the *beginning* of the interval.

3. Starting at the first interval, go step by step through all the intervals, calculating the influence of the constant rate (for that interval) on the integral.

4. If these are the relevant variables:

---

[1]By the way, the set of rules given here is only one of several ways to do this. This method of integration is called *Euler's Method*, and is probably the most well-known numerical integration algorithm.

$$d_{new} = \text{The value of the integral at the end of the interval}$$

$$d_{old} = \text{The value of the integral at the beginning of the interval}$$

$$v = \text{The constant rate of change for that interval}$$

$$dt = \text{the change in time over the interval}$$

Each interval, you need to calculate the new value of the integral (d_{new}) based upon the old value (d_{old}) plus the change in the integral.

Fill the formula for the *change* in the integral below:

$$d_{new} = \ d_{old} +$$

Question: How can the accuracy of this integration method be improved?

III. Using a spreadsheet to do numerical integration

A spreadsheet is a 2 dimensional grid of *cells* that allows you to create tables of numbers, formulas and text. It allows you to easily perform repeated calculations, and graph the results.

You should always label entries in your spreadsheet. This helps you to keep track of what the numbers and formulas mean.

A typical screen you might see when doing integration with a spreadsheet might be this:

	1	2	3	4	5
A	Time				
B	0.0	0.5	1.0	1.5	2.0
C					
D	Dt				
E		0.5	0.5	0.5	0.5
F	Rate				
G	30.0	35.0	40.0	45.0	50.0
H					
I	Integral				
J	0.0	15.0	32.5	52.5	77.5

In the above portion of the spreadsheet, the under the words "Time" and "Rate" were entered by hand. The numbers under the words "Dt" (delta time – the time interval), and "Integral" were calculated with formulas.

Note that there is no value for Dt in cell E1. Why might this be?

What would be the formula in cell E2?

What would be entered in cell J1? Why is this a number, not a formula?

[ ]

What would be the formula in cell J2?

[ ]

What would be the formula in cell J3?

[ ]

Note that once you enter a formula in a cell, you can copy similar formulas to the right using the **fill right** command. The spreadsheet will automatically adjust the formula so that it references the correct cells. (For example, if your formula for cell J2 calculated something based upon G1, cell J3 would use cell G2 instead.)

<div style="text-align:center">*Integration Problem-Solving Exercises 1*</div>

Back to the Future

You sit up, suddenly, surrounded by bright lights. You look around, confused - there are people in white coats all around you. You here someone say, "Welcome to the 24th century," and you fade out of conciousness.

When you wake up, you learn that you have been transported, using a highly experimental technique, from the (long-ago) year 1992. You are on a starship, the U.S.S. Cambridge, half-way to Alpha Centauri, four light-years from Earth. The ship is commanded by Captain Smith, a fierce looking bearded man who looks like he needs a pipe and a wooden leg.

You have been brought to this ship because you are desperately needed. You see, the Cambridge's computer, (affectionately called Eddie), has been struck by a random ninth dimensional quantum spear. (The captain tries to explain how it happened but you lose him after the first quark.) The important thing is that the Cambridge's navigational unit is only partly functioning. Some information needs to be computed by hand, and everyone in the 24th century has become so reliant on computers, they have forgotten their math.

Your task, Captain Smith explains, is to apply your mathematical skills and calculate a timetable for their journey.

"And we'll even pay you 15 glachmas an hour. If you work 2 days at 10 hours a day, that's umm, umm." Captain Smith pulls out his pocket calculator and begins punching buttons furiously.

"Never mind," you say magnanimously, "I accept your challenge regardless of how much I get paid – as long as it's a lot!"

The Problem

While the navigational system is severely damaged, some information is still available. Backup logs of planned periods of activation for the warp drive is brought to you. You realize that this represents the planned *acceleration* for the ship.

"Ah-ha!" you say. "We can do integration!" You quickly explain that integration is the process of adding up the acceleration to calculate the velocity, and adding up the velocity to calculate the ship's position.

The captain scratches his head in puzzlement. "Whatever you say. Just remember, we need to know the position of the ship at *every* point along the voyage (or at least, many points)."

Unfortunately, you do not have the final version of the logs. You have several different ones, each representing a different proposal for travel. The captain tells you to calculate each one, then select the plan that will get you to Alpha Centauri (4 light year away)) in the shortest time.

Cruise Control

Before you begin, you discover that the Cambridge is a hi-technology starship, and has the 24th century version of cruise control. Cruise control allows the helm computer to specify a desired velocity for the ship. The navigational computer will automatically set the correct accelerations. You find several sets of velocity programs still intact.

For the following sets of velocities, calculate the distance of the ship from it's starting point using a spreadsheet or calculator. Assume that at time 0, the ship was orbiting Earth at distance zero. Draw rough graphs below.
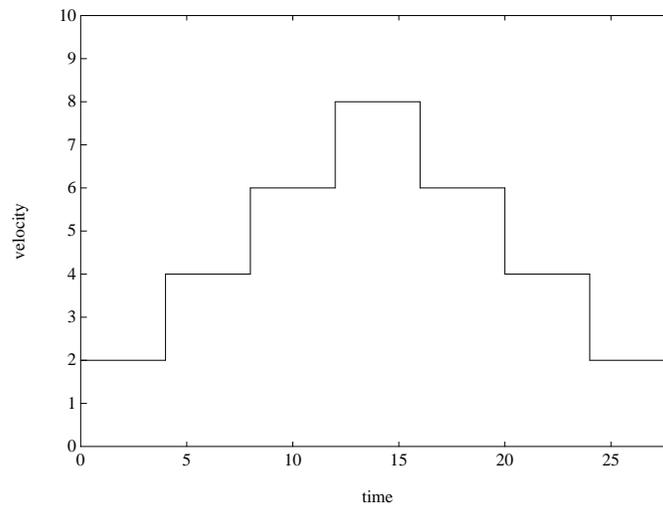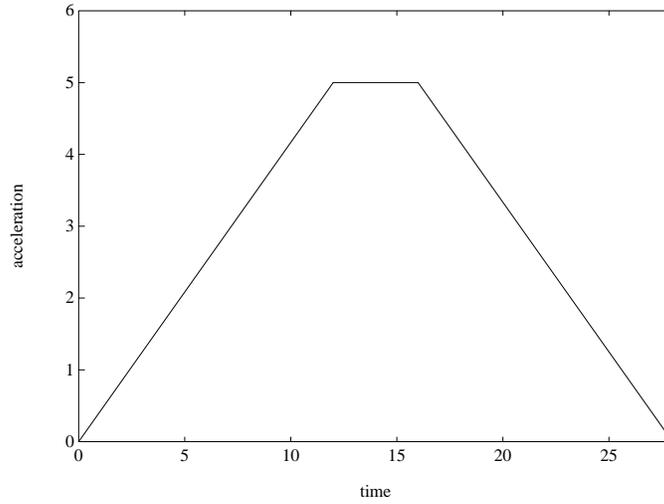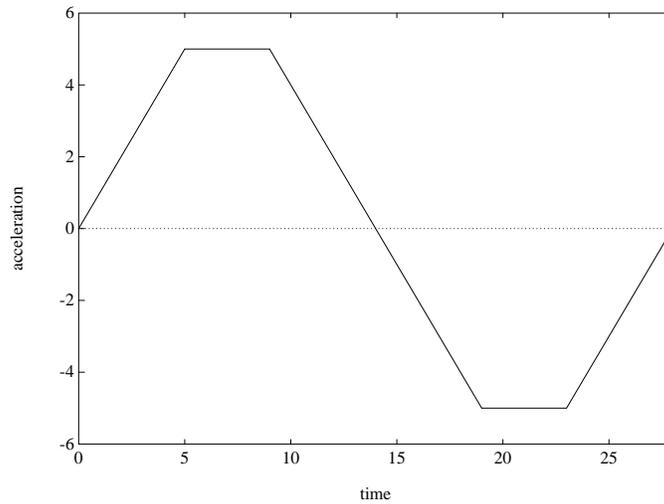

1.
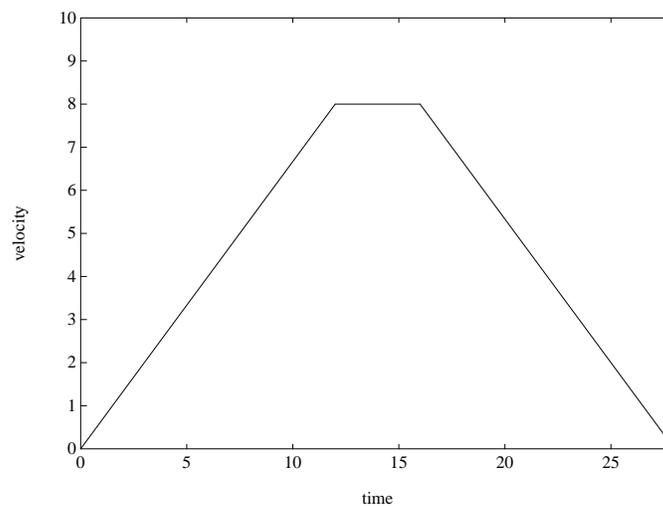
2.

Engine Control

For the following accelerations of the Cambridge, calculate the velocities and then the positions of the Cambridge using a spreadsheet or calculator.

1.

2.

Methods of Integration I: Numerical Integration

Integration Problem-Solving Exercises 2

Your assignment for today is to break into groups of two or three, and work with a computer or calculator and graph paper to answer *one* of the following questions. Please organize your work neatly with graphs and explanations. Be prepared to give a short (5 minute) presentation to the other groups on your findings.

1. Numerical integration is a powerful tool, but it is useful to have rules that will allow us to calculate integrals directly. For example, you already should know the rule to integrate constant functions of the form $f(t) = c$ (where c is a constant). Can you come up with a rule for equations of the form $f(t) = ct$? You may not even need a spreadsheet for that one. How about $f(t) = ct^2$ (much tougher). *After you figure out these rules*, you may want to try to determine the **power rule** by coming up with a formula to integrate $f(t) = ct^n$ (where both c *and* n are constants).

2. The method of integration we have learned in class is not the only one. Can you think of an algorithm that gives higher accuracy? Compare the error of your new method with the error of Euler's method while integrating $f(t) = t^2$. How does this error vary as the dt varies? How does the *difference* between Euler's method and your new method change as dt changes? Can you think of any other methods of numerical integration?

3. Several of you made comments in a previous class that lowering the dt (time interval) made the (approximate) numerical integration more accurate. How much more accurate is it? Determine a way to measure the accumulated error in integrating $f(t) = t^2$. You may ask me for the exact integral to compare. Integrate this function with a number of different dt's and compare the accuracy. Graph the accumulated error versus the size of the dt.

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